Tackling Uncertainty in Coalitional Games

JEAN-PAUL DOIGNON

Université Libre de Bruxelles

Based on the IJCAI(2022) paper with

MELTEM ÖZTÜRK and STEFANO MORETTI

Université Paris Dauphine

15th International Conference on Scalable Uncertainty Management (SUM2022)
(paper and slides are available)

Supported by French National Research Agency under THEMIS ANR-20-CE23-0018

Example

Marks for the joint homeworks of three students a, b and c:

$$
\begin{array}{rclcrcl}
\varnothing & \mapsto & 0 & \qquad & \{c\} & \mapsto & 3 \\
\{a\} & \mapsto & 5 & & \{a,c\} & \mapsto & 4 \\
\{b\} & \mapsto & 2 & & \{b,c\} & \mapsto & 3 \\
\{a,b\} & \mapsto & 6 & & \{a,b,c\} & \mapsto & 8
\end{array}
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How to assign individual notes? What if marks are uncertain?

Given a **worth** for each **coalition** of economic **players**,

- ▻ how to define **player scores**? (many proposals known);
- ▻ what if the coalitions worths are uncertain?

Example

For player c, say, compute the numerical differences when c joins:

$$\begin{array}{rr|r} v(\varnothing) = 0 & v(\{c\}) = 3 & v(\{c\}) - v(\varnothing) = \ \ 3 \\ v(\{a\}) = 5 & v(\{a,c\}) = 4 & v(\{a,c\}) - v(\{a\}) = -1 \\ v(\{b\}) = 2 & v(\{b,c\}) = 3 & v(\{b,c\}) - v(\{b\}) = \ \ 1 \\ v(\{a,b\}) = 6 & v(\{a,b,c\}) = 8 & v(\{a,b,c\}) - v(\{a,b\}) = \ \ 2 \end{array}$$

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BANZHAF (1965) PENROSE (1946) COLEMAN (1971) take the mean of the differences:

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SHAPLEY (1953) (& SHUBIK, 1954) the mean of the means for fixed sizes:

$$\pi_c^{\mathrm{Sha}}(v) = \frac{1}{3}\left(\frac{1}{1}\,3 + \frac{1}{2}\,(-1 + 1) + \frac{1}{1}\,2\right) = \frac{5}{3}$$

Similar for students a and b.

(see Banzhaf biography on wikipedia!)

Definition

A **set function** $v : 2^N \to \mathbb{R}$ is a **game** when $v(\varnothing) = 0$;

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For any given v, a **(regular) probabilistic scoring** π

(LUCCHETTI, MORETTI and PATRONE, 2015)

assigns to **player** i in N the **score**

$$\pi_i(v) = \sum_{S \in 2^N:\ i \notin S} p_i(S) \Big(v(S \cup \{i\}) - v(S) \Big)$$

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Any probabilistic scoring π is

linear: $\pi_i(\lambda\, v + \mu\, w) \quad = \quad \lambda\, \pi_i(v) + \mu\, \pi_i(w)$

steady: $\pi_i(\mathbf{1}) \quad = \quad \pi_j(\mathbf{1})$

(for all players i, j in N).

A hierarchy of scorings:

scorings which are linear and steady

probabilistic scorings $\quad p_i(S)$

$$\pi_i(v) \; = \sum_{S \in 2^N, i \notin S} p_i(S) \left(v(S \cup \{i\}) - v(S) \right)$$

semivalues $\quad q(|S|)$

$\frac{1}{2^{n-1}}$ Banzhaf scoring

Shapley scoring $\frac{1}{n \binom{n-1}{|S|}}$

What if worths are uncertain?

If we do not know the worths $v(A)$ exactly,

we might nevertheless know how v **ranks** the coalitions:

$$v(A_1) \geqslant v(A_2) \geqslant \ldots \geqslant v(A_{2^n})$$

Can we then infer how the scores **rank** the players?

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Given v and π, there can arise unstability or stability.

(ranking = weak order = total preorder)

Example

Two set functions v and w on $N = \{a, b, c\}$:

coalitions S	N	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{c\}$	$\{b\}$	$\{a\}$	$\varnothing$
worths $v(S)$	9	8	6	4	3	2	1	0
worths $w(S)$	9	8	7	6	5	3	1	0

The Banzhaf scores are

players i	a	b	c
scores $\pi_i^{\text{Ban}}(v)$	$15/4$	$14/4$	$11/4$
scores $\pi_i^{\text{Ban}}(w)$	$11/4$	$13/4$	$15/4$

Note that v and w rank the coalitions in the same way,
but the Banzhaf scores are in reversed order.

Definition

Two set functions $v, w : 2^N \to \mathbb{R}$ are **ordinally equivalent** when they rank the coalitions in the same way: for all S, T in 2^N,

$$v(S) \geqslant v(T) \iff w(S) \geqslant w(T)$$

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Exercise

Let π be any linear and steady scoring.

For any **bivalued** set function v and any players i, j, **stability** holds:

$$\pi_i(v) \leqslant \pi_j(v)$$

$$\iff$$

for all set functions u ordinally equivalent to v

$$\pi_i(u) \leqslant \pi_j(u)$$

For the rest of the talk, let

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i and j be two players in N.

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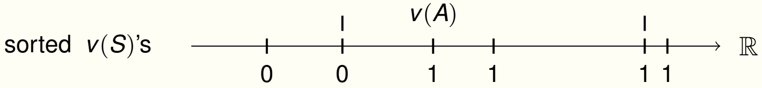
For any coalition A, set $v_A(S) = \begin{cases} 1 & \text{if } v(A) \leqslant v(S), \\ 0 & \text{otherwise.} \end{cases}$



We thus specify $2^{|N|}$ **binary** set functions v_A.

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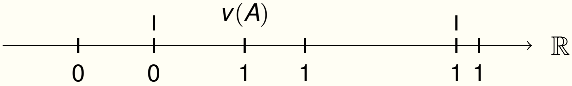
sorted $v(S)$'s



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Theorem

Assuming the scoring π on N is steady and linear:

for all set functions u ordinally equivalent to v: $\pi_i(u) \leqslant \pi_j(u)$

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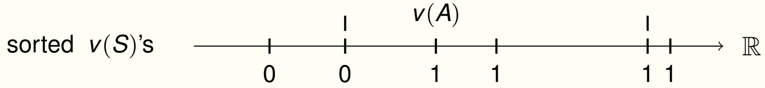

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Assuming the scoring π on N is steady and linear:

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$$\Longleftrightarrow$$

for all coalitions A in 2^N: $\pi_i(v_A) \leqslant \pi_j(v_A)$.

Theorem

Consider $v: 2^N \to \mathbb{R}$, two players i, j, and the Banzhaf scoring. Then:

for all set functions u ordinally equivalent to v: $\pi_i^{\text{Ban}}(u) \leqslant \pi_j^{\text{Ban}}(u)$

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$$\iff$$

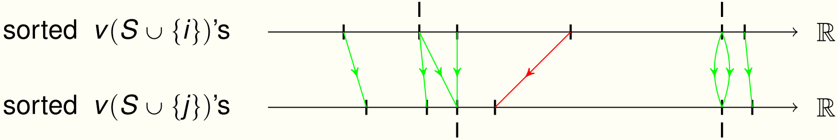
for $S \in 2^{N \setminus \{i,j\}}$, sort the numbers $v(S \cup \{i\})$, and the numbers $v(S \cup \{j\})$;

for each $k = 1, 2, \ldots, 2^{n-2}$:

the k-th smallest number $v(S \cup \{i\})$

is less or equal than

the k-th smallest number $v(S \cup \{j\})$.



An extension of previous theorem

from Banzhaf scorings to semivalues:

Theorem

Let π be a semivalue with parameter vector q.

Assume $\pi_i(v) \leqslant \pi_j(v)$ for $v : 2^N \to \mathbb{R}$, and two players i, j.

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$$\iff$$

$$\forall S \in 2^{N \setminus \{i,j\}} : \quad 0 \leqslant D_{i,j}^{\pi^{(q)}}(v, S).$$

Here $D_{i,j}^{\pi^{(q)}}(v, S_0)$ is some quantity involving

the parameters $q(k)$ of the semivalue π

(see paper or additional slide).

Future Work

(a) Design "efficient" algorithms,
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(b) Replace $v: 2^N \to \mathbb{R}$ with $v: \mathcal{S} \to \mathbb{R}$, where $\mathcal{S} \subseteq 2^N$.
Consult for instance (and its references)
Bilbao, Jiménez-Losada and Ordóñez (2019).

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(c) Extension to “stochastic games”?

Here $v(S)$ becomes a random variable, see for instance

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- Any nonempty coalitions of the above problems.

Thanks for your attention.

Definition of $D^{\pi^{(q)}}_{i,j}(v, S)$

For the given semivalue with parameters $q(k)$, set

$$q'(k) \;=\; q(k) + q(k+1).$$

For any S in $2^{N\setminus\{i,j\}}$, define successively

$$\mathcal{T}^{+}_{i,j} \;=\; \{T \in 2^{N\setminus\{i,j\}} \,|\, v(T \cup \{i\}) \leqslant v(S \cup \{j\}) \leqslant v(T \cup \{j\})\},$$

$$\mathcal{T}^{-}_{i,j} \;=\; \{U \in 2^{N\setminus\{i,j\}} \,|\, v(U \cup \{j\}) \leqslant v(S \cup \{j\}) v(U \cup \{i\})\},$$

$$D^{\pi^{(q)}}_{i,j}(v, S) \;=\; \sum_{T \in \mathcal{T}^{+}_{i,j}} q'(|T|) \;-\; \sum_{U \in \mathcal{T}^{-}_{i,j}} q'(|U|).$$

The quantity $D^{\pi^{(q)}}_{i,j}(u, S)$ takes the same value for all set functions u ordinally equivalent to v.