A capacity-based semantics for inconsistency-tolerant inferences

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Motivation

Two sources of uncertainty

- Incomplete information
- Inconsistency

Classical logic deals with incomplete information:

- Given a consistent knowledge base, a proposition is known to be true, known to be false, or unknown.
- These epistemic statuses can be captured
  - in modal logic ($\Box p$, $\Box\neg p$, $\neg\Box p \wedge \neg\Box\neg p$)
  - possibility theory ($N(p) = 1, N(\neg p) = 1, N(p) = N(\neg p) = 0$).

But classical logic cannot deal with inconsistency non-trivially:
what can be an inconsistent-tolerant semantics?

Motivation

In the presence of inconsistency

- The usual model-based semantic inference collapses: you cannot evaluate inconsistent knowledge bases on interpretations since $Mod(K) = \emptyset$

The way out: Extend the epistemic semantics of classical logic:

- Evaluate formulas on epistemic states $E \subseteq \mathcal{I}$ (non-empty subset of interpretations)
- p is known to be true in E iff $E \subseteq Mod(p)$, i.e., $N_E(p) = 1$
- This semantics is equivalent to the one of classical logic.

In the case of inconsistency, move to more general set functions beyond necessity measures

From necessity measures to capacities

Boolean necessity measures

- If the epistemic state E: $N(p) = 1$ if $E \subseteq Mod(p)$, and 0 otherwise.
- $N(A \cap B) = \min(N(A), N(B))$
- If $E = Mod(K)$, then $Cons(K) = \{p : N(Mod(p)) = 1\}$.

Boolean capacities

- A set function $g : 2^{\mathcal{I}} \to \{0, 1\}$ monotonic with inclusion.
- The family $\{A : g(A) = 1\}$ has minimal elements $\mathcal{F}_g$ forming an antichain of focal sets that determine g.
- $g(A) = 1$ iff $\exists E \in \mathcal{F}_g : E \subseteq A$.

A simple inconsistency-tolerant inference: $\models_{eit}$

Idea

inconsistency derives from the presence of conflicting sources of information

$$K \models_{eit} p \iff \exists p_i \in K, p_i \text{ consistent, such that } p_i \models p.$$

- Each consistent formula is supposed to come from a specific source of information: inconsistent sources ruled out.
- We do not allow for fusion of information from distinct sources: we only collect the available pieces of information (in the spirit of Belnap).
- Logical consequences $\mathbb{C}_{eit}(K) = \bigcup_{p_i \in K} \mathbb{C}_{PrL}(\{p_i\})$

A simple inconsistency-tolerant inference: $\models_{eit}$

Given an inconsistency-tolerant inference relation $\vdash_I$, is there a capacity g such that $K \vdash p$ if and only if $g(Mod(p)) = 1$?

Capacity associated to K under $\models_{eit}$: $g_K = \max_{p_i \in K} N_i$,

Remarks

- Focal sets: $\mathcal{F}_K = \{[p_i] : p_i \in K, \nexists q \in K, q \models p_i\}$.
- $K \models_{eit} p \iff g_K([p]) = 1$.
- If K is consistent the **eit** inference is weaker than classical inference.
- Modus Ponens is not a valid inference rule

Strengthening $\models_{eit}$

We can refine the capacity g_K into $g_{K\exists}$ asking that

$$\forall C \subseteq K \text{ consistent}, g_{K\exists}([\wedge_{p_i \in C} p_i]) = \min_{p_i \in C} g_{K\exists}(Mod(p_i)) = 1$$

Inference: $K \models_{\exists} p \iff g_{K\exists}([p]) = 1 \iff \{q_1, \ldots, q_k\} \models_{eit} p$

Remarks

- **Disjoint focal sets**: conjunctions q_k of formulas in the maximal consistent subsets $MC_k, k = 1, \ldots, m$.
- We cross-fertilize the pieces of information in K
- If K is consistent this inference comes down to classical inference.
- This is the existential (or weak) consequence of Rescher and Manor (1970) $\models_{\exists}$ based on maximal consistent subsets.

The 4 epistemic statuses of a proposition

The status of a proposition p wrt the inconsistent knowledge base K, using the *eit*-inference can be defined: p is

- *supported* if $K \models_{eit} p$ and $K \not\models_{eit} \neg p$;
- *rejected* if $K \models_{eit} \neg p$ and $K \not\models_{eit} p$;
- *unknown* if p is neither supported nor rejected, i.e., $K \not\models_{eit} p$ and $K \not\models_{eit} \neg p$;
- *conflicting* if p is both supported and rejected, i.e., $K \models_{eit} p$ and $K \models_{eit} \neg p$.

The four epistemic statuses can be expressed by means of g_K, letting $A = [p]$:

- *Support*: $g_K(A) = 1$ and $g_K(A^c) = 0$. *Rejection*: $g_K(A^c) = 1$ and $g_K(A) = 0$.
- *Ignorance*: $g_K(A) = g_K(A^c) = 0$. *Conflict*: $g_K(A) = g_K(A^c) = 1$.

Relations with Belnap logic

The four pairs $(g_K(A), g_K(A^c)) = \{(0,0),(1,0),(0,1),(1,1)\}$ encode the 4 Belnap epistemic truth-values (NONE,TRUE, FALSE, BOTH) and form a bilattice.

Belnap setting and logic

- Sources i express their knowledge about atomic propositions $a \in V$: $t_i(a) = 1, 0$ or unknown
- $T_i = \{a \in V : t_i(a) = 1\}$, $F_i = \{a \in V : t_i(a) = 0\}$.
- define $K_B = \{p_1, \ldots p_n\}$ where $p_i = (\bigwedge_{a \in T_i} a) \wedge (\bigwedge_{b \in F_i} \neg b)$.

Relations with Belnap logic

- The Belnap epistemic statuses of each atomic proposition can be retrieved using inference $\models_{eit}$. p is
  - TRUE if $K_B \models_{eit} p$ and $K_B \not\models_{eit} \neg p$;
  - FALSE if $K_B \models_{eit} \neg p$ and $K_B \not\models_{eit} p$;
  - NONE if p is neither supported nor rejected, i.e., $K_B \not\models_{eit} p$ and $K_B \not\models_{eit} \neg p$;
  - BOTH if p is both supported and rejected, i.e., $K_B \models_{eit} p$ and $K_B \models_{eit} \neg p$.
- The epistemic statuses of composite propositions can be obtained by truth tables.
- Belnap logic can be captured by an elementary modal logic with capacity semantics (Ciucci and Dubois, IJAR, 2019).

Other inconsistency-tolerant logics with capacity semantics

- Priest logic of Paradox
  - Belnap logic without truth-value NONE
  - need capacities such that $\max(g(A), g(A^c)) = 1$, typically possibility measures.
- Argumentative inference:
  - p follows from K if p follows classically from a consistent subset of K but its negation does not.
  - $K \vdash_A p$ if and only if $K \vdash_\exists p$ and $K \not\vdash_\exists \neg p$ (using Rescher and Manor existential inference).
  - So $K \vdash_A p$ if and only if $g(Mod(p)) = 1$ and $g(Mod(\neg p)) = 0$ (Belnap TRUE).
  - It is not truth-functional.

Conclusion

- We have proposed a capacity-based semantics to reasoning under inconsistency
- capacity semantics cover a number of old approaches
- Other approaches could perhaps be covered: quasi-classical and other paraconsistent logics.
- towards a unified semantic view of inconsistency-tolerant inference
- potential bridge to valued uncertainty theories and logic (probability and beyond)