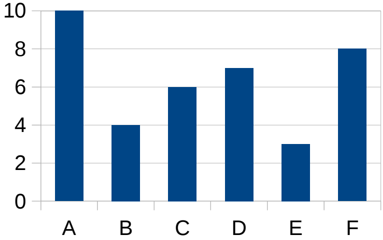
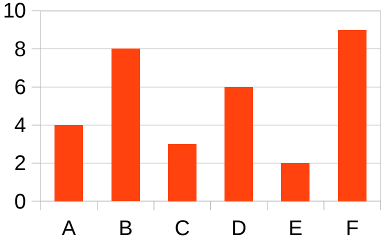
# An introduction to robust combinatorial optimization

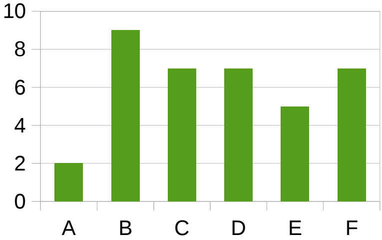
Michael Poss

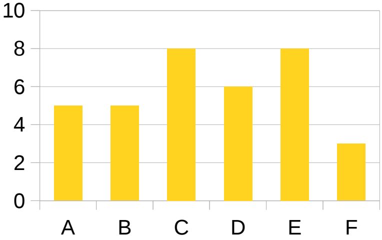
October 18, 2022

# Optimality: criterion max min (or min max)



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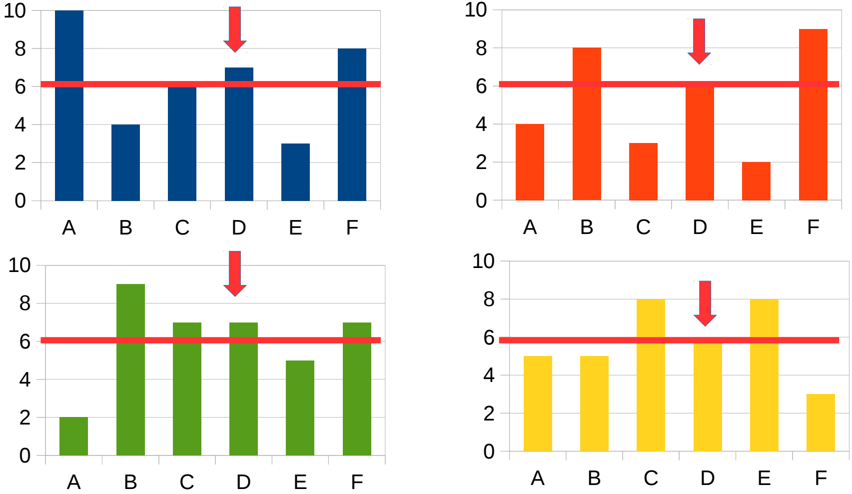
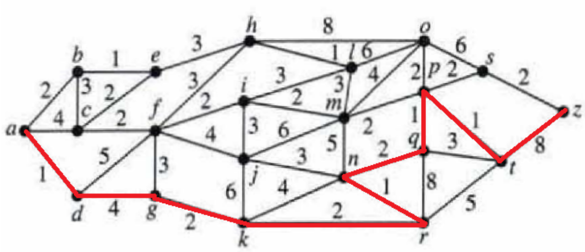


# Feasibility: all scenarios matter




Like shortest path but with 2 resources

- Cost
- Time $\leq\ C \Leftrightarrow \sum_{a \in p} u_a \leq C \ \forall u \in \mathcal{U}$

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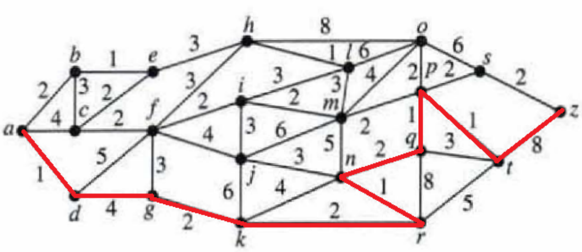


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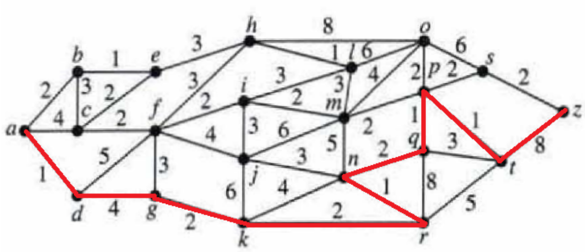

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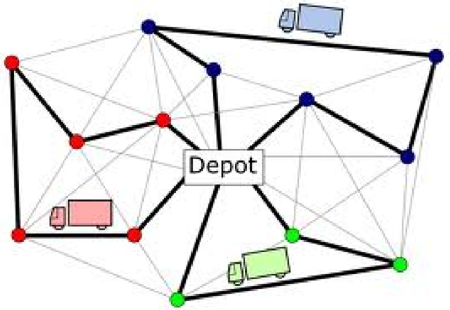
# Vehicle routing problem




Different robust counterparts:

- Cost uncertainty
- Demand uncertainty:

$$\sum_{i \in \text{route}} u_i \leq Capacity, \quad \forall u \in \mathcal{U}, route \in Routes$$

- Travel time uncertainty

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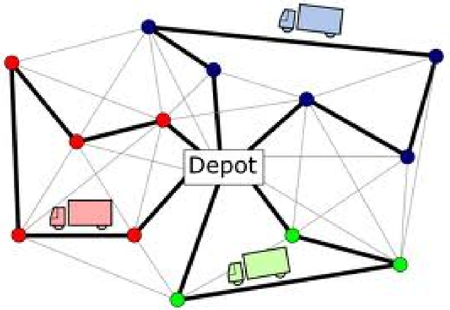


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# Numerical example with demand uncertainty

- A company needs to be pick up packages of uncertain dimensions
- The company owns 6 vehicles
- Possibility of renting an additional vehicle
- Simulating the failure probability by sampling $10^6$ demand values

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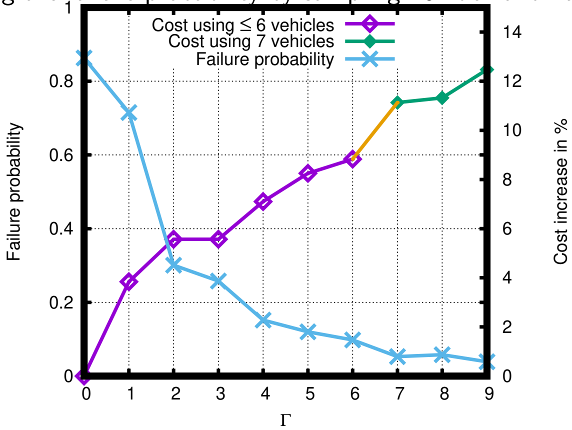
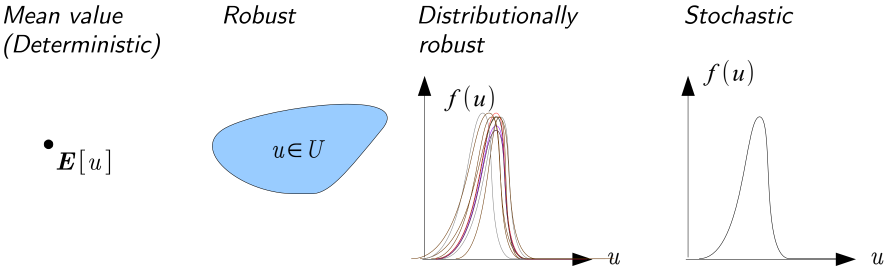


**How much do we know ?**



# static VS adjustable

| | |
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| Static | decisions --→ uncertainty revealed |
| *Complexity* | Easy for LP ☺, $\mathcal{NP}$-hard for combinatorial optimization ☹ |
| | MILP reformulation ☺ |
| Two-stages | decisions --→ uncertainty revealed --→ more decisions |
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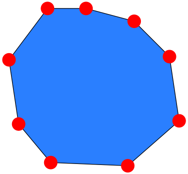
# discrete uncertainty VS convex uncertainty

$\mathcal{U} = \text{vertices}(\mathcal{P})$



**Observation**

In many cases, $\mathcal{U} \sim \mathcal{P}$.

Exceptions:

- robust constraints $f(x, u) \leq b$ and $f$ non-concave in $u$
- multi-stages problems with integer adjustable variables

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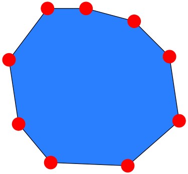

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# Robust combinatorial optimization

**Combinatorial problem**

- $\mathcal{X} \subseteq \{0,1\}^n$, $u_0 \in \mathbb{R}^n$

$$CO \qquad \min_{x \in \mathcal{X}} u_0^T x.$$

**Robust counterparts with cost uncertainty**

1. $\mathcal{X} \subseteq \{0,1\}^n, \mathcal{U} \subset \mathbb{R}^n$

$$\mathcal{U}\text{-}CO \qquad \min_{x \in \mathcal{X}} \max_{u \in \mathcal{U}} u_0^T x \quad \Leftrightarrow \quad \min_{x \in \mathcal{X}} f(x)$$

2. Regret version:

$$\overbrace{\min_{x \in \mathcal{X}} \max_{u \in \mathcal{U}} \left( u_0^T x - \min_{y \in \mathcal{X}} u_0^T y \right)}^{f(x)}$$

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# General robust counterpart

$\mathcal{X} = \mathcal{X}^{comb} \cap \mathcal{X}^{num}$ :

$\mathcal{X}^{comb}$ Combinatorial nature, **known**.

$\mathcal{X}^{num}$ Numerical uncertainty: $u_j^T x \leq b_j,\ j = 1, \dots, m$, **uncertain**.

## Example (Vehicle routing)

$\mathcal{X}^{comb}$ routes in the graph

$\mathcal{X}^{num}$ demand cannot exceed the capacity

## Robust counterpart

$$\mathcal{U}\text{-}CO \qquad \min \left\{ \begin{array}{ll} \max_{u_0 \in \mathcal{U}_0} u_0^T x \ : & \\ u_j^T x \leq b_j, \quad j = 1, \dots, m, \ u_j \in \mathcal{U}_j, & \\ x \in \mathcal{X}^{comb} & \end{array} \right\}. \qquad \begin{array}{r} (1) \\ (2) \\ \end{array}$$

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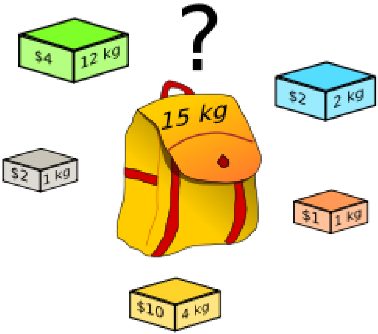
# Knapsack problem




How to maximize profit without violating the knapsack capacity?

# Deterministic situation

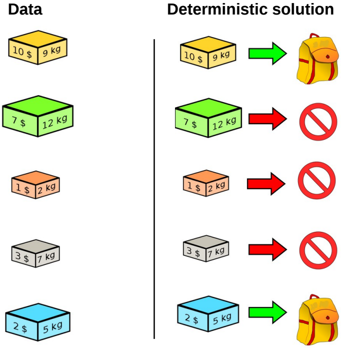
profit $p$, weight $u$, capacity $C$

$$\begin{aligned} \max \quad & \sum_{i \in N} p_i x_i \\ \text{s.t.} \quad & \sum_{i \in N} u_i x_i \leq C \\ & x \in \{0,1\}^{|N|}. \end{aligned}$$

- $\mathcal{NP}$-hard in the weak sense: difficult, but not too much.
- Arizes in vehicle routing, facility location, network design, assignement problems, investment problems, ...

# Uncertainty

- 5 items, capacity $= 15$
- Scenario $k \Rightarrow u_k \times \frac{4}{3}$

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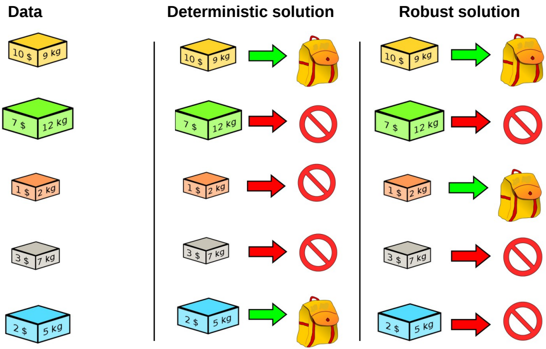


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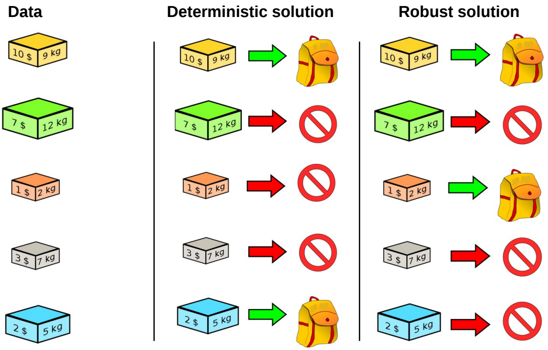


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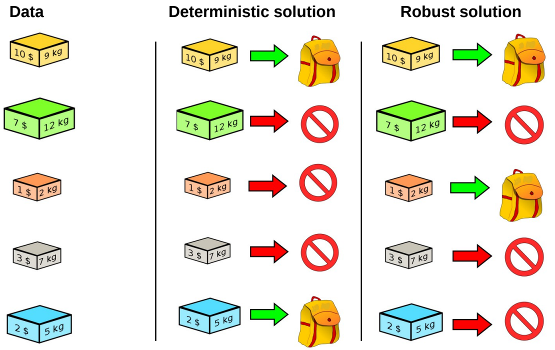


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# discrete uncertainty: $\mathcal{U}$-$CO$ is hard [Kouvelis and Yu, 2013]

**Theorem**

*The robust shortest path, assignment, spanning tree, ... are $\mathcal{NP}$-hard even when $|\mathcal{U}| = 2$.*

**Proof.**

1. SELECTION PROBLEM: $\min_{S \subseteq N, |S|=p} \sum_{i \in S} u_i$
2. ROBUST SEL. PROB.: $\min_{S \subseteq N, |S|=p} \max_{u \in \mathcal{U}} \sum_{i \in S} u_i$
3. PARTITION PROBLEM: $\min_{S \subseteq N, |S|=|N|/2} \max \left( \sum_{i \in S} a_i, \sum_{i \in N \setminus S} a_i \right)$
4. Reduction: $p = \frac{|N|}{2}$, and $\mathcal{U} = \{u^1, u^2\}$ such that

$$u_i^1 = a_i \qquad \text{and} \qquad u_i^2 = \frac{2}{|N|} \sum_k a_k - a_i$$

$$\Rightarrow \quad \max_{u \in \mathcal{U}} \sum_{i \in S} u_i = \max \left( \sum_{i \in S} a_i, \sum_{i \in N \setminus S} a_i \right)$$

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1. SELECTION PROBLEM: $\min_{S \subseteq N, |S|=p} \sum_{i \in S} u_i$
2. ROBUST SEL. PROB.: $\min_{S \subseteq N, |S|=p} \max_{u \in \mathcal{U}} \sum_{i \in S} u_i$
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4. Reduction: $p = \frac{|N|}{2}$, and $\mathcal{U} = \{u^1, u^2\}$ such that

$$u_i^1 = a_i \quad \text{and} \quad u_i^2 = \frac{2}{|N|} \sum_k a_k - a_i$$

$$\Rightarrow \quad \max_{u \in \mathcal{U}} \sum_{i \in S} u_i = \max \left( \sum_{i \in S} a_i, \sum_{i \in N \setminus S} a_i \right)$$

# discrete uncertainty: $\mathcal{U}$-$CO$ is hard [Kouvelis and Yu, 2013]

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# polyhedral uncertainty: $\mathcal{U}$-$CO$ is still hard (but solvable)

**Theorem (Kouvelis and Yu [2013])**

*The robust shortest path, assignment, spanning tree, ... are $\mathcal{NP}$-hard even when $\mathcal{U}$ has a compact description.*

**Proof.**

1. $u^T x \leq b, \quad u \in \mathcal{U} \quad \Leftrightarrow \quad u^T x \leq b, \quad u \in \text{ext}(\mathcal{U})$
2. $\mathcal{U} = \text{conv}(u^1, u^2) \Rightarrow n$ equalities and 2 inequalities

□

**Theorem (Dualization - Ben-Tal and Nemirovski [1998])**

*Problem $\mathcal{U}$-$CO$ is equivalent to a mixed-integer linear program.*

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# Dualization - cost uncertainty

## Theorem (Ben-Tal and Nemirovski [1998])

*Consider* $\alpha \in \mathbb{R}^{l \times n}$ *and* $\beta \in \mathbb{R}^l$ *that define polytope*

$$\mathcal{U} := \{u \in \mathbb{R}^n_+ \ : \ \alpha_k^T u \leq \beta_k, \ k = 1, \ldots, l\}.$$

*Problem* $\min_{x \in \mathcal{X}} \max_{u \in \mathcal{U}} u^T x$ *is equivalent to a compact MILP.*

## Proof.

Dualizing the inner maximization:

$$\min_{x \in \mathcal{X}} \max_{u \in \mathcal{U}} u^T x = \min_{x \in \mathcal{X}} \min \left\{ \sum_{k=1}^{l} \beta_k z_k \ : \sum_{k=1}^{l} \alpha_{ki} z_k \geq x_i, i = 1, \ldots, n, z \geq 0 \right\}$$

$$= \min \left\{ \sum_{k=1}^{l} \beta_k z_k \ : \sum_{k=1}^{l} \alpha_{ki} z_k \geq x_i, i = 1, \ldots, n, z \geq 0, x \in X \right\}$$

# Dualization example

Can also be applied to robust constraints!

**Example (Static problem)**

$$\begin{aligned} \max \quad & 3x_1 + 5x_2 + 9x_3 \\ \text{s.t.} \quad & u_1x_1 + u_2x_2 + u_3x_3 \leq 8 \ \forall u \in \mathcal{U} \\ & x \in \{0,1\}^3. \end{aligned}$$

**Example (Uncertainty polytope)**

$$\mathcal{U} \equiv \left\{ \begin{array}{ll} 3u_1 + u_2 + u_3 \leq 10 & [z_1] \\ u_1 + 2u_2 \leq 8 & [z_2] \\ u_1 + 2u_3 \leq 7 & [z_3] \\ u_2 + u_3 \leq 5 & [z_4] \\ u_1, u_2, u_3 \geq 0 & \end{array} \right\}$$

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The static problem is equivalent to:

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# Cutting plane algorithms [Bertsimas et al., 2016]

$\mathcal{U}_0^* \subset \mathcal{U}_0$, $\mathcal{U}_j^* \subset \mathcal{U}_j$

## Master problem

$$
MP \qquad \min \left\{ \begin{array}{ll} z : & \\ u_j^T x \leq b_j, & j = 1, \ldots, m, \; u_j \in \mathcal{U}_j^*, \\ u_0^T x \leq z, & u_0 \in \mathcal{U}_0^*, \\ a_k^T x \leq d_k, & k = 1, \ldots, \ell \\ x \in \{0,1\}^n & \end{array} \right\}
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1. **Solve** $MP \rightarrow$ get $\tilde{x}, \tilde{z}$
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# Simpler structure: $\mathcal{U}^\Gamma$-robust combinatorial optimization

- $\mathcal{U}$ = vertices($\mathcal{P}$): good, but need "simpler" $\mathcal{P}$

$$\mathcal{U}^\Gamma = \left\{ \overline{u}_i \leq u_i \leq \overline{u}_i + \hat{u}_i, i = 1, \ldots, n, \sum_{i=1}^{n} \frac{u_i - \overline{u}_i}{\hat{u}_i} \leq \right\}$$

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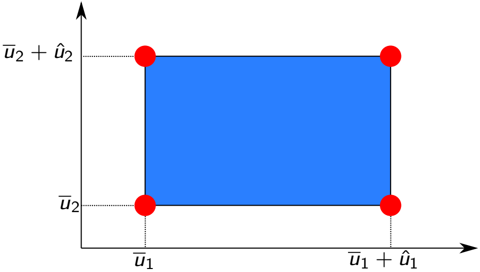


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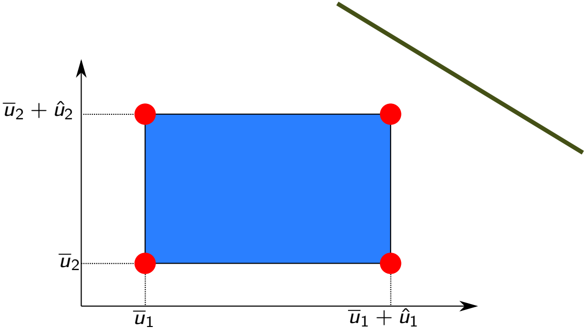


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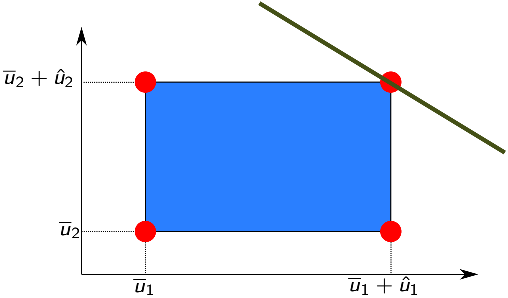


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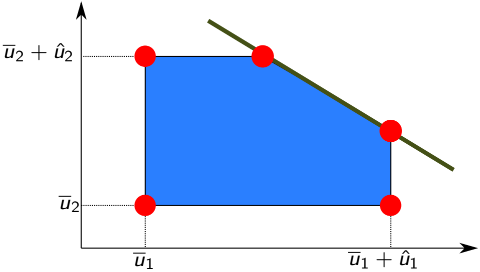


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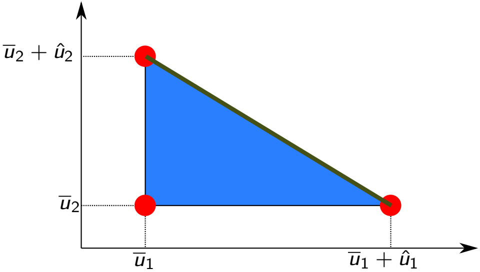


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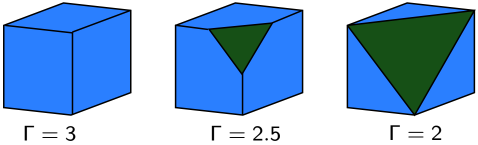
# Iterative algorithms for $\mathcal{U}^\Gamma$

$$\mathcal{P} = \left\{ \overline{u}_i \leq u_i \leq \overline{u}_i + \hat{u}_i, i = 1, \ldots, n, \sum_{i=1}^{n} \frac{u_i - \overline{u}_i}{\hat{u}_i} \leq \Gamma \right\}$$




**Theorem (Bertsimas and Sim [2003], Goetzmann et al. [2011], Álvarez-Miranda et al. [2013], Lee and Kwon [2014])**

*Cost uncertainty* $\mathcal{U}^\Gamma$-*CO* $\Rightarrow$ *solving* $\sim n+1$ *problems CO.*

*Numerical uncertainty* $\mathcal{U}^\Gamma$-*CO* $\Rightarrow$ *solving* $\sim (n+1)^m$ *problems CO.*

# Iterative algorithms for $\mathcal{U}^\Gamma$

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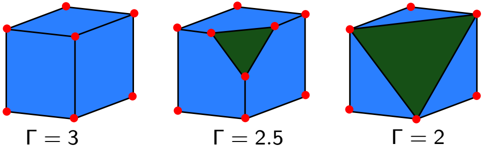


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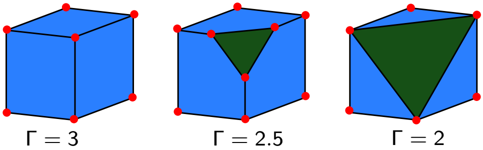


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# Example

$$\Gamma \hat{u}_\ell + \min_{x \in \mathcal{X}} \sum_i (\overline{u}_i + [\hat{u}_i - \hat{u}_\ell]^+) x_i$$

**Example (Static problem)**

$$\begin{aligned} \max \quad & 3x_1 + 5x_2 + 9x_3 \\ \text{s.t.} \quad & u_1 x_1 + u_2 x_2 + u_3 x_3 \le 8 \ \ \forall u \in \mathcal{U}^\Gamma \\ & x \in \{0,1\}^3. \end{aligned}$$

**Example (Uncertainty polytope)**

$$\mathcal{U}^\Gamma = \left\{ \begin{array}{l} 3 \le u_1 \le 3 + 1 \\ 2 \le u_2 \le 2 + 2 \\ 1 \le u_3 \le 1 + 4 \\ \frac{u_1 - 3}{1} + \frac{u_2 - 2}{2} + \frac{u_3 - 1}{4} \le 1 \end{array} \right\}$$

**Example (Solution algorithm)**

Solve 4 knapsack problems

$$(\hat{u}_\ell = 0) \quad \begin{aligned} \max \quad & 3x_1 + 5x_2 + 9x_3 \\ \text{s.t.} \quad & 4x_1 + 4x_2 + 5x_3 \le 8 \\ & x \in \{0,1\}^3. \end{aligned}$$

$$(\hat{u}_\ell = 1) \quad \begin{aligned} \max \quad & 3x_1 + 5x_2 + 9x_3 \\ \text{s.t.} \quad & 3x_1 + 3x_2 + 4x_3 \le 7 \\ & x \in \{0,1\}^3. \end{aligned}$$

$$(\hat{u}_\ell = 2) \quad \begin{aligned} \max \quad & 3x_1 + 5x_2 + 9x_3 \\ \text{s.t.} \quad & 3x_1 + 2x_2 + 3x_3 \le 6 \\ & x \in \{0,1\}^3. \end{aligned}$$

$$(\hat{u}_\ell = 4) \quad \begin{aligned} \max \quad & 3x_1 + 5x_2 + 9x_3 \\ \text{s.t.} \quad & 3x_1 + 2x_2 + 1x_3 \le 5 \\ & x \in \{0,1\}^3. \end{aligned}$$

# Example

$$\Gamma \hat{u}_\ell + \min_{x \in \mathcal{X}} \sum_i (\overline{u}_i + [\hat{u}_i - \hat{u}_\ell]^+) x_i$$

## Example (Static problem)

$$\begin{aligned} \max \quad & 3x_1 + 5x_2 + 9x_3 \\ \text{s.t.} \quad & u_1 x_1 + u_2 x_2 + u_3 x_3 \le 8 \ \ \forall u \in \mathcal{U}^\Gamma \\ & x \in \{0,1\}^3. \end{aligned}$$

## Example (Uncertainty polytope)

$$\mathcal{U}^\Gamma \equiv \left\{ \begin{array}{l} 3 \le u_1 \le 3 + 1 \\ 2 \le u_2 \le 2 + 2 \\ 1 \le u_3 \le 1 + 4 \\ \frac{u_1 - 3}{1} + \frac{u_2 - 2}{2} + \frac{u_3 - 1}{4} \le 1 \end{array} \right\}$$

## Example (Solution algorithm)

Solve 4 knapsack problems

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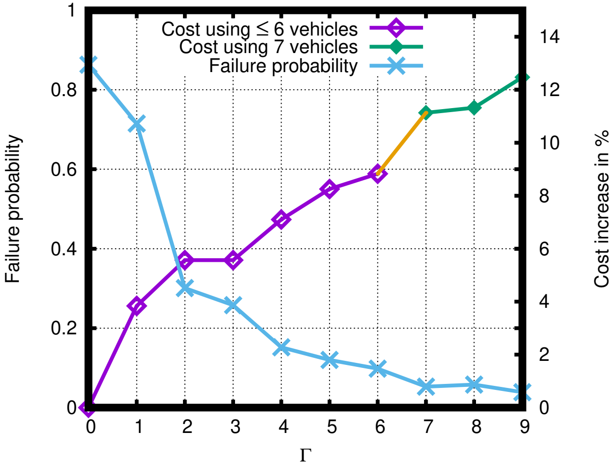
# $\mathcal{U}^{\Gamma}$: example

- Need to specify $\overline{u}$, $\hat{u}$, and $\Gamma$
- Example: $\overline{u} = \mu$ and $\hat{u} = \sigma$



# Vehicle Routing Problem (CVRP) - Compact formulation

$x_{ij}^k$ vehicle $k$ uses arc $(i,j)$?
$u_i$ uncertain demand at node $i$

$$
\begin{aligned}
\min \quad & \sum_{i,j} c_{ij} x_{ij}^k \\
\text{s.t.} \quad & \textit{flow conservation} \\
& \textit{cycle-breaking} \\
& \sum_{i,j} u_i x_{ij}^k \leq C, \quad \forall k \in K, u \in U \\
& x \text{ binary}
\end{aligned}
$$

## Dualization

$$
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\text{s.t.} \quad & \textit{flow conservation} \\
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& \Gamma z^k + \sum_i y_i^k \leq C, \quad \forall i \in V, k \in K \\
& z^k + y_i^k \geq \sum_j x_{ij}^k, \quad \forall k \in K \\
& y, z \geq 0 \\
& x \text{ binary}
\end{aligned}
$$

## Iterative algorithm

$|K|$ capacity constraints $\Rightarrow (n+1)^{|K|}$ nominal problems to be solved!

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# Vehicle Routing Problem (CVRP) - Set-partition

$x_r^k$ vehicle $k$ uses route $r$?

$$\begin{aligned} \min \quad & \sum_{r,k} c_r x_r^k \\ \text{s.t.} \quad & \sum_{r:i\in r} x_r^k = 1, \quad \forall i \in V \\ & x \text{ binary} \end{aligned}$$

## Pricing problem

$x_{ij}$ new route uses arc $(i,j)$?
$u_i$ uncertain demand at node $i$

$$\begin{aligned} \min \quad & \sum_{i,j} \kappa_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{i,j} u_i x_{ij} \le C, \quad \forall u \in U \\ & x \textit{ is a route} \end{aligned}$$

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$$\begin{aligned} \min \quad & \sum_{i,j} \kappa_{ij} x_{ij} \\ \text{s.t.} \quad & \Gamma z + \sum_i y_i \le C \\ & z + y_i \ge \sum_j x_{ij}, \quad \forall i \in V \\ & y, z \ge 0 \\ & x \textit{ is a route} \end{aligned}$$

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Only 1 capacity constraint $\Rightarrow n+1$ nominal problems to be solved!

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# Examples of numerical results (CVRP)

| In. | # | Iterative algo | | | Dualization and strengthening | | |
|---|---|---|---|---|---|---|---|
| cls | in. | #n. | t. | #opt. | gap | t. | #opt. |
| A | 26 | 1.00 | 2.91 | 26 | 1.97% | 3440.31 | 12 |
| B | 23 | 1.05 | 5.98 | 23 | 1.39% | 250.96 | 13 |
| E | 11 | 1.00 | 11.40 | 11 | 2.19% | 573.01 | 5 |
| F | 3 | 5.37 | 833.42 | 2 | 1.10% | 55.76 | 2 |
| M | 3 | 3.33 | 153.51 | 3 | 2.70% | 86700.00 | 1 |
| P | 24 | 1.00 | 1.48 | 24 | 2.09% | 976.36 | 10 |
| all | 90 | 1.11 | 4.75 | 89 | 1.87% | 981.90 | 43 |

# Are all problems easy?

Hard problems must have one of

1. non-constant number of robust “linear” constraints
2. “non-linear” constraints/cost function

**Theorem (Pessoa et al. [2015])**

*$\mathcal{U}^\Gamma$-robust shortest path with time windows is $\mathcal{NP}$-hard in the strong sense.*

**Theorem (Bougeret et al. [2016])**

*Minimizing the weighted sum of completion times is $\mathcal{NP}$-hard in the strong sense.*

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# Cookbook for static problems

## Dualization

good easy to apply

bad breaks combinatorial structure (e.g. shortest path)

## Cutting plane algorithms (branch-and-cut)

good handle non-linear functions

bad implementation effort

## Iterative algorithms

good good theoretical bounds

bad solving $n^s$ problems can be too much

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# Additional tutorials

- Decision rules (multi-stage setting): Ayse Nur Arslan (Roadef 2022)
  Available on youtube
- Modelling: Boris Detienne (Roadef 2020)
  Available on video.umontpellier.fr

# Open Journal of Mathematical Optimization (OJMO)

- With classical publishers, either
  - papers are behind an (expensive) paywall;
  - or authors pay ($\pm$2k) for Open Access (the so-called **gold** OA)
- OJMO provides a **free** OA alternative (thanks to Mersenne)
- Papers have doi, indexed in Scopus, DBLP, zbMATH, Crossref, ...

Area Editors

- **Continuous Optimization** - David Russell Luke
- **Discrete Optimization** - Sebastian Pokutta
- **Optimization under Uncertainty** - Guzin Bayraksan
- **Computational aspects and applications** - Michael Poss

17 published papers, 13 under review, 73 submissions ... and one prize!

2021 Beale — Orchard-Hays Prize Citation

**Alberto Costa and Giacomo Nannicini**
"RBFOpt: an open-source library for black-box optimization with costly function evaluations"
Mathematical Programming Computation 10 (2018) 597-629.
"On the implementation of a global optimization method for mixed-variable problems"
Open Journal of Mathematical Optimization 2 (2021).

**Visit https://ojmo.centre-mersenne.org/**

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