Selecting the Most Relevant Elements from a Ranking over Sets

Sébastien Konieczny[2], Stefano Moretti[1], **Ariane Ravier[1]** & Paolo Viappiani[1]

[1] LAMSADE, Paris Dauphine
[2] CRIL, Université d'Artois

15th International Conference on Scalable Uncertainty Management (SUM 2022)

October 18th, 2022

THEMIΣ

Introduction

An introductory example

An introductory example

Framework

A population

A set of coalitions from the population

Lex-cel

Let $\mathcal{P}(N)$ denote the set of all possible subsets one can build from a given population N.
Let $\mathcal{R}(X)$ denote the set of rankings over a given set X.
Let $\succ$ be a power relation of the form $\Sigma_1 \succ \Sigma_2 \succ \ldots \succ \Sigma_k$.
Let $i_k = |\{S \in \Sigma_k : i \in S\}|$, and $\theta^{\succeq}(i)$ be a k-dimensional vector such that $\theta^{\succeq}(i) = (i_1, i_2, \ldots, i_k)$.

Lexicographic excellence

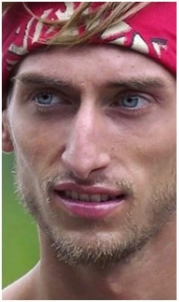
The *lexicographic excellence (lex-cel)* is the binary relation $R_{le}^{\succeq}$ such that for all $\succeq \in \mathcal{R}(\mathcal{P}(N))$ and all $i, j \in N$:

$$i \; R_{le}^{\succeq} \; j \Longleftrightarrow \theta^{\succeq}(i) \quad \geq_L \quad \theta^{\succeq}(j),$$

with $\geq_L$ the lexicographic order.

Power relation

Lex-cel



$\mathcal{P}$

$\mathcal{P}$

Why not the entire ranking?

▷ Difficulties computing the ranking

Why not the entire ranking?

- ▷ Difficulties computing the ranking
- ▷ Only interested in the winner(s)

Coalitional Social Choice Function

Coalitional Social Choice Function

Let $\mathcal{P}(N)$ denote the set of all possible subsets one can build from a given population N.
Let $\mathcal{R}(X)$ denote the set of rankings over a given set X.

Coalitional Social Choice Function

A *coalitional social choice function* is a map

$$B : \mathcal{R}(\mathcal{P}(N)) \to \mathcal{P}(N)$$

which associates to each power relation $\succeq \in \mathcal{R}(\mathcal{P}(N))$ a non-empty subset $B(\succeq) \in \mathcal{P}(N)$ which is interpreted as the set of *winners* in $\succeq$.

Desirable properties

All-Indifferent-All-Winners

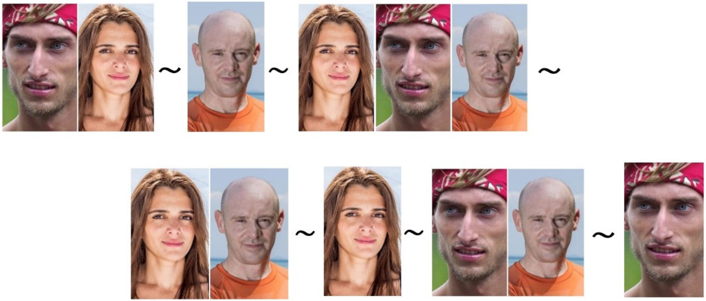
Axiom 1 (AIAW)

Consider a power relation $\succeq \in \mathcal{R}$ such that for all $S, T \in \mathcal{P}(N)$, it holds that

$$S \sim T,$$

then a coalitional social choice function B satisfies the property All-Indifferent-All-Winners if it holds that $B(\succeq) = N$.

All-Indifferent-All-Winners



All-Indifferent-All-Winners

$B(\succ) =$ 

Monotonicity for Winners

Axiom 2 (MW)

Consider two power relations $\succeq, \sqsupseteq \in \mathcal{R}(\mathcal{P}(N))$ and their respective quotient orders $\succ$ and $\sqsupset$ such that:

- $\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \Sigma_l$,
- $\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \Sigma_{l-1} \sqsupset \Sigma \sqsupset \Sigma_l \setminus \Sigma$,
- with $\Sigma \subseteq \Sigma_l$.

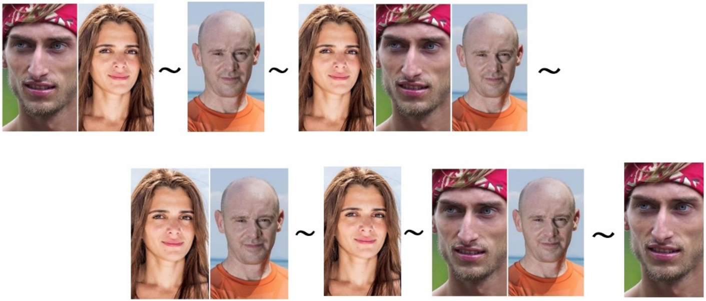
Take a coalitional social choice function B and let $T \subseteq B(\succeq)$ be the set of most represented winners over Σ, *i.e.*

$$T = \{i \in B(\succeq) : i_\Sigma \geq j_\Sigma \ \forall j \in B(\succeq)\}.$$

We say that B satisfies the Monotonicity for Winners property if it holds that

$$T \subseteq B(\sqsupseteq).$$

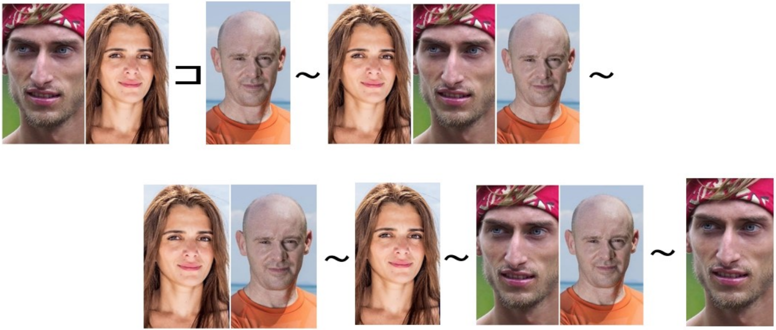
Monotonicity for Winners



Monotonicity for Winners

$B(\succ) =$ 

Monotonicity for Winners



Monotonicity for Winners

$\subseteq B\ (\sqsupset)$

Dominance

Axiom 3 (D)

Consider two power relations $\succeq, \sqsupseteq \in \mathcal{R}(\mathcal{P}(N))$ and their respective quotient orders $\succ$ and $\sqsupset$ such that:

- $\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \Sigma_l$,
- $\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \Sigma_{l-1} \sqsupset \Sigma \sqsupset \Sigma_l \setminus \Sigma$,
- with $\Sigma \subsetneq \Sigma_l$.

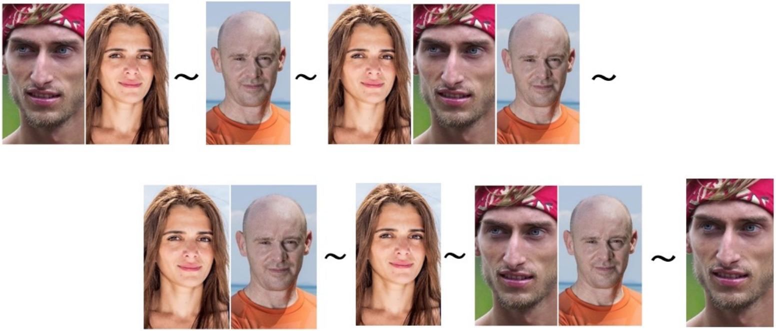
Take a cscf B and let $L \subseteq B(\succeq)$ be the set of winners that are strictly less represented than other winners over Σ, *i.e.*

$$L = \{j \in B(\succeq) : \exists i \in B(\succeq) \text{ with } i_\Sigma > j_\Sigma\}.$$

We say that B satisfies the *dominance* property if it holds that

$$B(\sqsupseteq) \subseteq N \setminus L.$$

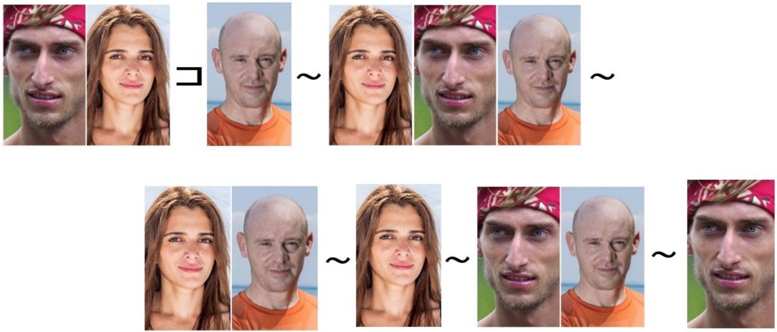
Dominance



Dominance

$B(\succ) =$ 

Dominance



Dominance

 $\subseteq B\,(\sqsupset)$

Dominance

$\subseteq B(\sqsupset)$

$\notin B(\sqsupset)$

Independence for Losers from the Worst Set

Axiom 4 (ILWS)

A cscf B satisfies the property of Independence for Losers from the Worst Set if $\forall \succeq \in \mathcal{R}(\mathcal{P}(N))$ with the associated quotient order $\succ$ such that

$$\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \Sigma_l$$

and $i \in N$ such that $i \notin B(\succeq)$, then for any partition $T_1, \ldots, T_m$ of Σ_l and for any power relation $\sqsupseteq \in \mathcal{R}(\mathcal{P}(N))$ with the associated quotient order $\sqsupset$ such that

$$\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \Sigma_{l-1} \sqsupset T_1 \sqsupset \cdots \sqsupset T_m,$$

it holds that $i \notin B(\sqsupseteq)$.

Lex-cel-based coalitional social choice function

The *lex-cel* coalitional social choice function

Lex-cel coalitional social choice function

Let $\succeq \in \mathcal{R}(\mathcal{P}(N))$. The *lex-cel coalitional social choice function* is the map $B_{le} : \mathcal{R}(\mathcal{P}(N)) \to \mathcal{P}(N)$ such that for all $\succeq \in \mathcal{R}(\mathcal{P}(N))$:

$$B_{le}(\succeq) = \{i \in N : i \; R_{le}^{\succeq} \; j, \forall j \in N\}.$$

The *lex-cel* cscf and our presented axioms

Proposition

The All-Indifferent-All-Winners, Monotonicity for Winners, Dominance and Independence for Losers from the Worst Set axioms are logically independent.

Theorem

The coalitional social choice function B_{le} is the unique solution fulfilling Axioms AIAW, M, D and ILWS.

Conclusion and perspectives

Conclusion and perspectives

Four logically independent axioms representing desirable properties.

We have introduced the *lex-cel* coalitional social choice function and proven that it is the only one to satisfy these four axioms.

Conclusion and perspectives

Four logically independent axioms representing desirable properties.

We have introduced the *lex-cel* coalitional social choice function and proven that it is the only one to satisfy these four axioms.

More uncertainty: what about given an incomplete order over the subsets of our population ?