An approach to inconsistency-tolerant reasoning about probability based on Łukasiewicz logic

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Motivation

To advocate:

(i) Two-layer fuzzy modal logics are conceptually simple logics to reason about uncertainty, in particular FP(Ł), based on Łukasiewicz fuzzy logic Ł, is specially appropriate for probabilistic reasoning

To show:

(ii) FP(RPL) = FP(Ł) + truth-constants provides suitable tools to reason under inconsistent probabilistic information

Outline

- Lukasiewicz fuzzy logic, Rational Pavelka logic
- FP(RPL): a two-layer fuzzy modal system for probabilistic reasoning
- Measuring consistency in FP(RPL) theories
- Repairing probabilistic theories in FP(RPL)
- Conclusions

Łukasiewicz logic: Ł

Jan Łukasiewicz: three valued-logic to model future contingents (1920), then generalised to n-valued logic (1922), and to a $[0,1]$-valued logic with A. Tarski (1930)

Primitive connectives are $\to$ and $\neg$

AXIOMS:

- $\varphi \to (\psi \to \varphi)$
- $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
- $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
- $((\varphi \to \psi) \to \psi) \to (((\psi \to \varphi) \to \varphi)$

Algebraic semantics: variety of MV-algebras

Standard MV-algebra: $[0,1]_{MV} = ([0,1], \Rightarrow, \neg, 0, 1)$, with $x \Rightarrow y = \min(1, 1 - x + y), \neg x = 1 - x$

Łukasiewicz logic: Ł

Definable connectives:

$$\varphi \& \psi := \neg(\neg\varphi \oplus \neg\psi)$$
$$\varphi \oplus \psi := \neg\varphi \to \psi$$
$$\varphi \ominus \psi := \varphi \& \neg\psi$$
$$\varphi \wedge \psi := \varphi \& (\varphi \to \psi)$$
$$\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$$
$$\neg\varphi := \varphi \to \bar{0}$$

$[0, 1]_{MV}$

$$a \otimes b = \max(0, x + y - 1)$$
$$a \oplus b = \min(1, a + b)$$
$$a \ominus b = \max(0, a - b)$$
$$a \wedge b = \min(a, b)$$
$$a \vee b = \max(a, b)$$
$$\neg a = 1 - a$$

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$$\neg a = 1 - a$$

Standard completeness theorem: for every *finite* set of formulas $T \cup \{\varphi\}$,

$$T \vdash_{Ł} \varphi \text{ if, and only if, } T \models_{[0,1]_{MV}} \varphi$$

Expanding Ł with truth-constants: RPL

Łukasiewicz logic is mainly a qualitative fuzzy logic:

$$\vdash_{Ł} \varphi \rightarrow_{L} \psi \text{ when } e(\varphi) \leq e(\psi), \text{ for all } e$$

To explicitly reason about truth-degrees one can, e.g., introduce truth-constants into the language of Ł

$$truth(\varphi) \geq 0.6, truth(\psi) \geq 0.8 \models truth(\varphi \& \psi) \geq 0.4$$

Rational Pavelka logic (Pavelka, 79), (Hájek, 98): introduce a truth-constant $\bar{r}$ for every rational $r \in [0, 1]$

Rational Pavelka Logic

Axioms and rules of RPL are those of Ł plus the following countable set of book-keeping axioms for truth-constants:

$$(\mathrm{BK}_{\rightarrow}) \quad \bar{r} \rightarrow_L \bar{s} \equiv \overline{\min(1, 1 - r + s)}, \quad \text{for any } r, s \in [0,1]_{\mathbb{Q}}$$

Similar bookkepping axioms are derivable, for instance,

$$(\mathrm{BK}_{\&}) \quad \bar{r} \& \bar{s} \equiv \overline{\max(r + s - 1, 0)}, \quad \text{for any } r, s \in [0,1]_{\mathbb{Q}}$$

$$(\mathrm{BK}_{\neg}) \quad \neg_L \bar{r} \equiv \overline{1 - r}, \quad \text{for any } r \in [0,1]_{\mathbb{Q}}$$

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Standard completeness theorem: for every *finite* set of RPL-formulas $T \cup \{\varphi\}$,

$$T \vdash_{RPL} \varphi \text{ if, and only if, } T \models_{RPL} \varphi$$

Pavelka-style completeness: ...

A fuzzy (modal) approach to reason about uncertainty

After (Hájek-G-Esteva, 95; Hájek, 98):

- introduce a modality P, s.t. for each classical proposition φ,

  $\mathbf{P}\varphi$ reads e.g. "φ **is probable**"

- $P\varphi$ is a gradual, fuzzy proposition: the higher is the probability of φ, the truer is $P\varphi$

- intuitive semantics: for φ a two-valued, crisp proposition one can define e.g.

  $$truth(P\varphi) = probability(\varphi)$$

  (which is different from $truth(\varphi) = probability(\varphi)$!!!)

A fuzzy (modal) approach to reason about uncertainty

Observation: the language of Łukasiewicz is expressive enough to encode properties and computations with probability (and other measures), e.g.

$$\begin{aligned} Prob(A \cup B) &= Prob(A) + Prob(B) - Prob(A \cap B) \\ &= Prob(A) \oplus (Prob(B) \ominus Prob(A \cap B)) \end{aligned}$$

$$\begin{aligned} Nec(A \cap B) &= \min(Nec(A), Nec(B)) \\ &= Nec(A) \wedge Nec(B) \end{aligned}$$

Idea: axioms of different uncertainty measures on φ's to be encoded as *axioms of suitable fuzzy modal logic* theories over the $P\varphi$'s

$FP(Ł)$: A TWO-LEVEL FRAMEWORK

Probabilistic formulas — Łukasiewicz

$$P\varphi \equiv \overline{0.3},\quad P(\varphi \wedge \psi) \to_Ł P\chi\,,\quad \overline{0.6} \to_Ł P(\psi \vee \varphi), \ldots$$

....

Probabilistic atoms

$$P\varphi,\quad P(\varphi \wedge \psi \to \chi)\,,\quad P\neg(\psi \wedge \chi), \ldots$$

Events — CPC

$$\neg(\psi \wedge \chi),\quad \varphi \wedge \psi \to \chi,\quad \varphi \vee (\psi \to \chi), \ldots$$

$FP(Ł)$: A SIMPLE PROBABILITY LOGIC (HEG, 95), (HÁJEK, 98)

A two-level language:

(i) **Non-modal formulas:** $\varphi, \psi, \ldots$, built from variables using the classical logic connectives $\wedge$ and $\neg$. Set denoted by $\mathcal{L}$.

(ii) **Modal formulas:** $\Phi, \Psi, \ldots$, builtn from
- atomic modal formulas $P\varphi$, with $\varphi \in \mathcal{L}$
- compound, by combining them with Lukasiewicz connectives: $(\&_Ł, \rightarrow_Ł)$ and rational truth constants $\bar{r}$

Non wff formulas: $\varphi \rightarrow_Ł P\psi,\ P(P\varphi \wedge P\chi)$

FP(Ł): AXIOMATIZATION

- Axioms and rule of CPC for non-modal formulas
- Axioms of Łukasiewicz logic for modal formulas

- Probabilistic axioms:

(FP1) $P(\varphi \to \psi) \to_Ł (P\varphi \to_Ł P\psi)$

(FP2) $P(\varphi \vee \psi) \equiv (P\varphi \to_Ł P(\varphi \wedge \psi)) \to_Ł P\psi$

or equiv. $P(\varphi \vee \psi) \equiv P\varphi \oplus (P\psi \ominus P(\varphi \wedge \psi))$

(FP3) $P(\neg\varphi) \equiv \neg_Ł P\varphi$

- Deduction rules of FP(Ł) are *modus ponens* for $\to_Ł$ and
  (-) *necessitation* for P: from φ derive $P\varphi$

$FP(Ł)$: Semantics

Probabilistic Kripke models $M = (W, e, \mu)$

- $e : W \times Var \to \{0, 1\}$
- $\mu : \mathcal{U} \subseteq 2^W \to [0, 1]$ probability
- atomic modal formulas: $e_\mu(P\varphi) = \mu([\varphi])$
- compound modal formulas: $e_\mu(\Phi \star \Psi) = f_\star(e_\mu(\Phi), e_\mu(\Psi))$, for $\star \in \{\&_L, \to_L\}$

$M = (W, e, \mu)$ is a model of Φ if $e_\mu(\Phi) = 1$

Probabilistic logical consequence: $T \models_{FP} \Phi$

- μ can be regarded as a probability on classical formulas $\mu : \mathcal{L} \to [0, 1]$

Completeness: For any finite modal theory T:

$$T \vdash_{FP} \Phi \text{ iff } T \models_{FP} \Phi$$

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Remind: $FP(Ł)$ is mainly a qualitative probabilistic logic ...

But, everything smoothly extends if we add rational truth-constants to Ł: move from $FP(Ł)$ to $FP(RPL)$

Consistency degree of FP(RPL) theories

Let T be an inconsistent *finite* theory of FP(RPL), i.e.

$$[\![T]\!] = \{\mu \in \mathcal{P}(\mathcal{L}) \mid \text{for all } \Psi \in T, e_\mu(\Psi) = 1\} = \emptyset.$$

Let $\beta \in [0, 1]$. The set of *β-generalised models* of T:

$$[\![T]\!]_\beta = \{\mu \in \mathcal{P}(\mathcal{L}) \mid \text{for all } \Psi \in T, e_\mu(\Psi) \geq \beta\}.$$

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Definition: Consistency degree of T

$$Con(T) = \sup\{\beta \in [0, 1] \mid [\![T]\!]_\beta \neq \emptyset\}$$

$Inc(T) = 1 - Con(T)$

- T is inconsistent iff $Con(T) < 1$ iff $Inc(T) > 0$

Consistency degree of FP(RPL) theories

- $Con(T)$ is rational

and since, $e_\mu(\Phi) \geq \beta$ iff $e_\mu(\overline{\beta} \to_{Ł} \Phi) = 1$,

$$Con(T) = \max\{\beta \text{ rational } \mid T_\beta = \{\overline{\beta} \to \Phi \mid \Phi \in T\} \text{ is consist.}\}$$

$Con(T)$ is the minimal (global) weakening for T to become consistent

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- $Con(T) = \sup_\mu \bigwedge_{\Phi \in T} e_\mu(\Phi)$ $\quad$ $Inc(T) = \inf_\mu (1 - \bigwedge_{\Phi \in T} e_\mu(\Phi))$

That is, $Con(T)$ = maximal degree to which the probabilistic constraints logically expressed in T can be satisfied

$\Longrightarrow$ $Inc(T)$ is a (sort of) violation-based measure (Potyka, De Bona - Finger)

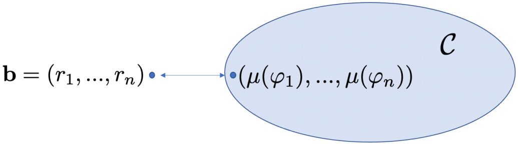
Consistency degree of FP(RPL) theories

The case of precise assignments: $KB = \{Pr(\varphi_i) = r_i\}_{i=1,\ldots,n}$

Represent KB as the theory $T = \{\overline{r_i} \equiv P\varphi_i\}_{i=1,\ldots,n}$ in FP(RPL)

$$1 - Con(T) = Inc(T) = \inf_{\mu} \bigvee_{i=1,\ldots,n} |\mu(\varphi_i) - r_i|$$

i.e. Chebyshev distance of $\mathbf{b} = (r_1, \ldots, r_n) \in [0,1]^n$ to the convex set $\mathcal{C} = \{(\mu(\varphi_1), \ldots, \mu(\varphi_n)) \in [0,1]^n\}_{\mu}$ of *consistent* prob. assignments



$Inc(T)$ is, in this case, a distance-based measure (Thimm, De Bona - Finger)

Consistency degree of FP(RPL) theories

For precise assignment theories, $Con(\cdot)$ fits with the frame of distance-based and violation-based inconsistency measures for (unconditional) probabilistic bases (De Bona-Finger-Potyka-Thimm, 2018)

- consistency,
- monotonicity,
- irrelevance of syntax,
- weak independence,
- continuity

but it does not seem to fit well with the so-called "fuzzy logic"-based inconsistency measures ...

Reasoning under inconsistency in FP(RPL)

Let $Con(T) = \alpha > 0$

Repair of T: weaken the theory by the consistency degree

$$T_\alpha = \{\overline{\alpha} \to_Ł \Phi \mid \Phi \in T\}$$

Example: let $\alpha = 0.8$ and $\Phi := \overline{0.6} \equiv P\varphi$, then

$$\begin{aligned} \overline{\alpha} \to_Ł \Phi \quad &:= (\overline{0.8} \otimes \overline{0.6} \to_Ł P\varphi) \wedge (P\varphi \to_Ł (\overline{0.8} \to_Ł \overline{0.6})) \\ &:= (\overline{0.4} \to_Ł P\varphi) \wedge (P\varphi \to_Ł \overline{0.8}) \end{aligned}$$

Inconsistent-tolerant inference:

$$\begin{aligned} T \mathrel{|\!\!\approx^*} \Phi \quad & \text{if} \quad T_\alpha \vdash_{FP(RPL)} \Phi \\ & \text{if} \quad e_\mu(\Phi) = 1 \text{ for all probabilities } \mu \in [\![T]\!]_\alpha. \end{aligned}$$

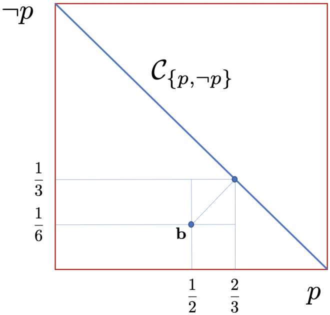
- $T \mathrel{|\!\!\approx^*} \bot$ only if $Con(T) = 0$.

Reasoning under inconsistency in FP(RPL)

Example: $T = \{P(p) \equiv \overline{1/2}, P(\neg p) \equiv \overline{1/6}\}$

$Con(T) = 1 - Incon(T) = 1 - 1/6 = 5/6$

$\mathbf{b} = (1/2, 1/6)$



Repaired theory: $T_{5/6} = \{\overline{5/6} \to (P(p) \equiv \overline{1/2}),\ \ \overline{5/6} \to (P(\neg p) \equiv \overline{1/6})\}$

$= \{\overline{1/3} \to P(p) \to \overline{2/3},\ \ \overline{0} \to P(\neg p) \to \overline{1/3}\}$

$T \mid\!\approx^* P(p) \equiv \overline{2/3}$

Example 2:

$T^+ = T \cup \{P(q) \equiv \overline{1/2}\} = \{P(p) \equiv \overline{1/2}, P(\neg p) \equiv \overline{1/6}, P(q) \equiv \overline{1/2}\}$

$Con(T^+) = Con(T) = 5/6$ (weak-independence)

$T^+_{5/6} = T_{5/6} \cup \{\overline{5/6} \rightarrow (P(q) \equiv \overline{1/2})\} = T_{5/6} \cup \{\overline{1/3} \rightarrow P(q) \rightarrow \overline{2/3}\}$

$T^+ \not\approx^* P(q) \equiv \overline{1/2}$

A too conservative repair ... in fact:

- $T^+ \approx^* \Phi$ iff, for all repairs S of T^+, $S \vdash_{FP(RPL)} \Phi$

A "LOCAL" REPAIR PROCEDURE

STEP 1 Let $Con(T^{>}) = \alpha_1$

- Identify minimal inconsistent $S \subseteq T$ such that $Con(S) = \alpha_1$.
  - $T^{=} = \bigcup\{S \subseteq T \mid S \text{ minimal such that } Con(S) = \alpha_1\} \neq \emptyset$
  - $T^{>} = T \setminus T^{=}$

  Weaken $T^{=}$: $\quad T^{(1)} = \{\overline{\alpha_1} \rightarrow \Phi \mid \Phi \in T^{=}\}$
- If $T^{>} = \emptyset$, then STOP and $T^{R} = T^{(1)}$

STEP 2 Let $Con(T^{>}) = \alpha_2 > \alpha_1$

- Identify minimal inconsistent $S \subseteq T^{>}$ such that $Con^{*}(S) = \alpha_2$.
  - $(T^{>})^{=} = \bigcup\{S \subseteq T^{>} \mid S \text{ minimal such that } Con^{*}(S) = \alpha_2\} \neq \emptyset$
  - $(T^{>})^{>} = T^{>} \setminus (T^{>})^{=}$

  Weaken $(T^{>})^{=}$: $\quad T^{(2)} = \{\overline{\alpha_2} \rightarrow \Phi \mid \Phi \in (T^{>})^{=}\}$
- If $(T^{>})^{>} = \emptyset$, then STOP and $T^{R} = T^{(1)} \cup T^{(2)}$

  ...

...

Example revisited

$T = \{P(p) \equiv \overline{1/2}, P(\neg p) \equiv \overline{1/6}, P(q) \equiv \overline{1/2}\}$

STEP 1: $Con(T) = 5/6$

- $T^{=} = \{P(p) \equiv \overline{1/2}, P(\neg p) \equiv \overline{1/6}\}$
- $T^{>} = \{P(q) \equiv \overline{1/2}\}$
- $T^{(1)} = \{\overline{5/6} \rightarrow (P(p) \equiv \overline{1/2}), \overline{5/6} \rightarrow (P(\neg p) \equiv \overline{1/6})\}$

STEP 2: $Con(T^{>}) = 1$

- $(T^{>})^{=} = \{P(q) \equiv \overline{1/2}\}$
- $(T^{>})^{>} = \emptyset$
- $T^{(2)} = \{P(q) \equiv \overline{1/2}\}$

$$T^R = T^{(1)} \cup T^{(2)}$$
$$= \{\overline{5/6} \rightarrow (P(p) \equiv \overline{1/2}), \overline{5/6} \rightarrow (P(\neg p) \equiv \overline{1/6}), P(q) \equiv \overline{1/2}\}$$

$$\approx^* P(p) \equiv \overline{2/3}, P(\neg p) \equiv \overline{1/3}, P(q) \equiv \overline{1/2}$$

Conclusions and future work

- Łukasiewicz fuzzy logic-based framework to measure and reasoning under inconsistent probabilistic information
- Can be cast in the frame of distance-based and violation-based measures

Future work:

- Generalize the frame to reason under conditional probabilistic information
  e.g. by moving from FP(RPL) to FP($Ł\Pi\frac{1}{2}$) fuzzy modal logic
- Generalize the approach for other classes of uncertainty models
  e.g., by moving from FP(RPL) to FN(RPL) fuzzy modal logic
  - (FN2) $N(\varphi \wedge \psi) \equiv N\varphi \wedge N\psi$

  $Con(T_{KB})$ = inconsistency level of a possibilistic KB à la D&P