Iterated Conditionals, Trivalent Logics, and Conditional Random Quantities

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Logical operations among conditionals

- Conditionals and compound conditionals are present in every day life. As an example of a conjoined conditional consider the outcomes of two covid tests.
- The sentence

  If the outcome of the first test is Positive, then the outcome of the second test is Positive

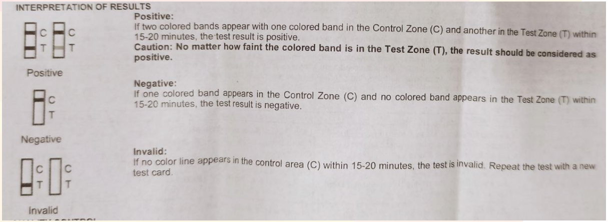
- should be read as the iterated conditional

  If the outcome of the first test is Positive, given that it is valid, then the outcome of the second test is Positive, given that it is valid,

  because both the antecedent and the consequent are conditionals.

Logical operations among conditionals

- For each given test the possible outcomes are *Positive* and *Negative*, given that the test is valid.



INTERPRETATION OF RESULTS

Positive

Positive:
If two colored bands appear with one colored band in the Control Zone (C) and another in the Test Zone (T) within 15-20 minutes, the test result is positive.
Caution: No matter how faint the colored band is in the Test Zone (T), the result should be considered as positive.

Negative

Negative:
If one colored band appears in the Control Zone (C) and no colored band appears in the Test Zone (T) within 15-20 minutes, the test result is negative.

Invalid

Invalid:
If no color line appears in the control area (C) within 15-20 minutes, the test is invalid. Repeat the test with a new test card.

- How to assign degree of beliefs to iterated conditionals?

Outline

- We recall some trivalent logics.
- We analyze, in the setting of trivalent logics, the iterated conditional introduced by de Finetti and by Calabrese.
- We recall the iterated conditional introduced by Gilio and Sanfilippo, which is defined by means of the following structure

$$\Box|\bigcirc = \Box \wedge \bigcirc + \mathbb{P}(\Box|\bigcirc)\overline{\bigcirc}.$$

- By using this structure and the logical operations in the trivalent logics, we introduce some notions of iterated conditionals , in the setting of conditional random quantities.
- We check, for all of them, the validity of basic logical an probabilistic properties.

Conditional events, three valued logics and conditional random quantities

The conditional event $A|H$, with $H \neq \emptyset$, is defined as a three-valued logical entity ([de Finetti(1936)])

$$A|H = \begin{cases} \text{True}, & \text{if} \quad A \wedge H = AH \quad \text{is true;} \\ \text{False}, & \text{if} \quad \overline{A}H \quad \text{is true;} \\ \text{Void}, & \text{if} \quad \overline{H} \quad \text{is true.} \end{cases}$$

Agreeing to the betting metaphor of the coherence framework, if you assess $p = P(A|H)$, it implies that you agree to pay p and to receive $(AH + p\overline{H})$, i.e.,

$$\text{to pay } p \quad \text{in order to receive} \begin{cases} 1, & \text{if } AH \text{ is true,} \\ 0, & \text{if } \overline{A}H \text{ is true,} \\ p, & \text{if } \overline{H} \text{ is true (called off).} \end{cases}$$

Then, the indicator of $A|H$ (denoted by the same symbol) is defined as the random quantity (see, e.g., [Coletti and Scozzafava(1999), Lad(1995), Gilio(1989)]), where $P(A|H) = p$

$$A|H = AH + p \cdot \overline{H} = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ 0, & \text{if } \overline{A}H \text{ is true,} \\ p, & \text{if } \overline{H} \text{ is true (called off).} \end{cases}$$

Trivalent logics and compound conditionals

Compound and iterated conditionals are defined, by the large part of authors, as suitable conditional events, that is as a three-valued objects.

However, when compound conditionals are defined as conditional events many basic logical and probabilistic properties are lost.

We recall some notions of conjunction and disjunction for two conditional events $A|H$ and $B|K$ in suitable trivalent logics.

Conjunction as a suitable conditional event

We recall four different notions of conjunction ([Ciucci and Dubois(2012)])

1. $(A|H) \wedge_K (B|K) = AHBK|(HK \vee \overline{A}H \vee \overline{B}K)$, Kleene-Lukasiewicz-Heyting-de Finetti conj.;
2. $(A|H) \wedge_L (B|K) = AHBK|(HK \vee \overline{A}\,\overline{B} \vee \overline{A}\,\overline{K} \vee \overline{B}\,\overline{H} \vee \overline{H}\,\overline{K})$, Lukasiewicz conj.;
3. $(A|H) \wedge_B (B|K) = AHBK|HK$, Bochvar internal conjunction;
4. $(A|H) \wedge_S (B|K) = (AH \vee \overline{H}) \wedge (BK \vee \overline{K})|(H \vee K)$, Sobocinski conj. (or quasi conjunction).

	C_h	$A\|H$	$B\|K$	$\wedge_K$	$\wedge_L$	$\wedge_B$	$\wedge_S$
C_1	$AHBK$	T	T	T	T	T	T
C_2	$AH\overline{B}K$	T	F	F	F	F	F
C_3	$AH\overline{K}$	T	V	V	V	V	T
C_4	$\overline{A}HBK$	F	T	F	F	F	F
C_5	$\overline{A}H\overline{B}K$	F	F	F	F	F	F
C_6	$\overline{A}H\,\overline{K}$	F	V	F	F	V	F
C_7	$\overline{H}BK$	V	T	V	V	V	T
C_8	$\overline{H}\,\overline{B}K$	V	F	F	F	V	F
C_0	$\overline{H}\,\overline{K}$	V	V	V	F	V	V

Based on De Morgan's law, we obtain the disjunctions $(\vee_K, \vee_L, \vee_B, \vee_S)$ associated with the previous list of conjunctions.

Calabrese's iterated conditional

Definition 1

Given any pair of conditional events $A|H$ and $B|K$, the iterated conditional $(B|K)|_C(A|H)$ is defined as

$$(B|K)|_C(A|H) = B|(K \wedge (\overline{H} \vee A)). \tag{1}$$

We recall that the notion of conjunction and disjunction of conditionals used by Calabrese coincide with $\wedge_S$ and $\vee_S$.

Properties for iterated conditionals

Logical and probabilistic properties for basic conditionals
(P1) $B|A = AB|A$;
(P2) $AB \subseteq B|A$, and $P(AB) \leq P(B|A)$;
(P3) $P(AB) = P(B|A)P(A)$ (*compound probability theorem*);
(P4) given two logical independent events A, B, with $P(A) = x$ and $P(B) = y$, the extension $\mu = P(B|A)$ is coherent if and only if $\mu \in [\mu', \mu'']$, where

$$\mu' = \begin{cases} \frac{\max\{x+y-1,0\}}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \quad , \quad \mu'' = \begin{cases} \frac{\min\{x,y\}}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

We check the validity of the properties above when replacing events A, B by conditional events $A|H, B|K$.

C_h	$(A\|H) \wedge_S (B\|K)$	$(B\|K)\|_C(A\|H)$	$[(A\|H) \wedge_S (B\|K)]\|_C(A\|H)$
$AHBK$	True	True	True
$AH\overline{B}K$	False	False	False
$AH\overline{K}$	True	Void	True
$\overline{A}HBK$	False	Void	Void
$\overline{A}H\overline{B}K$	False	Void	Void
$\overline{A}H\overline{K}$	False	Void	Void
$\overline{H}BK$	True	True	True
$\overline{H}\overline{B}K$	False	False	False
$\overline{H}\,\overline{K}$	Void	Void	Void

(P1) $(B|K)|_C(A|H) \neq [(A|H) \wedge_S (B|K)]|_C(A|H)$: see constituent $AH\overline{K}$;
(P2) $(A|H) \wedge_S (B|K) \nsubseteq (B|K)|_C(A|H)$: see constituent $AH\overline{K}$;
(P3) is not satisfied: see Theorem 2;
(P4) is not satisfied: see Theorem 3.

Iterated conditional of Calabrese: lower/upper bounds and Property (P3)

Theorem 2

A probability assessment $\mathcal{P} = (x, y, z)$ *on the family of conditional events* $\mathcal{F} = \{A|H, (B|K)|_C(A|H), (A|H) \wedge_S (B|K)\}$ *is coherent if and only if* $(x, y) \in [0, 1]^2$ *and* $z \in [z', z'']$*, where* $z' = xy$ *and* $z'' = max(x, y)$*.*

From Theorem 2, we observe that $z = xy$ is not the unique coherent extension of the conjunction $(A|H) \wedge_S (B|K)$. Indeed, it holds that

$$P(A|H)P[(B|K)|_C(A|H)] \leq P[(A|H) \wedge_S (B|K)].$$

In particular, the assessment $(1, 0, 1)$ on $\mathcal{F}$ is coherent and hence

$$P(A|H)P[(B|K)|_C(A|H)] = 0 < P[(A|H) \wedge_S (B|K)] = 1.$$

Of course the same assessment $(1, 0, 1)$ on $\{A, B|A, AB\}$ is not coherent.
Then, as it could be $P(A|H)P[(B|K)|_C(A|H)] < P[(A|H) \wedge_S (B|K)]$ the pair $(\wedge_S, |_C)$ does not satisfy property (P3).

Iterated conditional of Calabrese: lower/upper bounds and Property (P4)

Theorem 3

A probability assessment $\mathcal{P} = (x, y, z)$ *on* $\mathcal{F} = \{A|H, B|K, (B|K)|_C(A|H)\}$ *is coherent if and only if* $(x, y, z) \in [0, 1]^3$.

We observe that the probability propagation rule (property (P4)) is not valid for Calabrese's iterated conditional.

Indeed, from Theorem 3, any probability assessment (x, y, z) on $\mathcal{F} = \{A|H, B|K, (B|K)|_C(A|H)\}$, with $(x, y, z) \in [0, 1]^3$ is coherent.

For instance, the assessment $(1, 1, 0)$ on $\mathcal{F}$ is coherent, while it is not coherent on $\{A, B, B|A\}$.

Iterated conditional of de Finetti

Definition 4

Given any pair of conditional events $A|H$ and $B|K$, de Finetti iterated conditional, denoted by $(B|K)|_{df}(A|H)$, is defined as

$$(B|K)|_{df}(A|H) = B|(AHK). \tag{2}$$

We recall that the notion of conjunction and disjunction of conditionals introduced by de Finetti coincide with $\wedge_K$ and $\vee_K$.

Properties of de Finetti's iterated conditional

C_h	$(A\|H) \wedge_K (B\|K)$	$(B\|K)\|_{df}(A\|H)$	$[(A\|H) \wedge_K (B\|K)]\|_{df}(A\|H)$
$AHBK$	True	True	True
$AH\overline{B}K$	False	False	False
$AH\overline{K}$	Void	Void	Void
$\overline{A}HBK$	False	Void	Void
$\overline{A}H\overline{B}K$	False	Void	Void
$\overline{A}H\overline{K}$	False	Void	Void
$\overline{H}BK$	Void	Void	Void
$\overline{H}\overline{B}K$	False	Void	Void
$\overline{H}\,\overline{K}$	Void	Void	Void

(P1) is satisfied: $[(A|H) \wedge_K (B|K)]|_{df}(A|H) = (B|K)|_{df}(A|H)$;
(P2) is satisfied: see constituent $AHBK$ and $AH\overline{B}K$;
(P3) not satisfied: see Theorem 5;
(P4) not satisfied: see Theorem 6;

Iterated conditional of de Finetti: lower/upper bounds and Property (P3)

Theorem 5

A probability assessment $\mathcal{P} = (x, y, z)$ on the family of conditional events $\mathcal{F} = \{A|H, (B|K)|_{df}(A|H), (A|H) \wedge_K (B|K)\}$ is coherent if and only if $(x, y) \in [0,1]^2$ and $z \in [z', z'']$, where $z' = 0$ and $z'' = xy$.

As $z = xy$ is not the unique coherent extension of the conjunction $(A|H) \wedge_K (B|K)$, it holds that

$$P[(A|H) \wedge_K (B|K)] \neq P[(B|K)|_{df}(A|H)]P(A|H).$$

For example, the assessment $\mathcal{P} = (1, 1, 0)$ is coherent on $\mathcal{F}$ but not on $\{A, B|A, AB\}$ because
$P[(A|H) \wedge_K (B|K)] = 0 < P[(B|K)|_{df}(A|H)]P(A|H) = 1$.
Then, property (P3) is not satisfied by the pair $(\wedge_K, |_{df})$.

Iterated conditional of de Finetti: lower/upper bounds and Property (P4)

Theorem 6

A probability assessment $\mathcal{P} = (x, y, z)$ *on* $\mathcal{F} = \{A|H, B|K, (B|K)|_{df}(A|H)\}$ *is coherent if and only if* $(x, y, z) \in [0, 1]^3$.

We observe that the probability propagation rule (P4) is not valid for de Finetti's iterated conditional.

Indeed, from Theorem 6, any probability assessment (x, y, z) on $\mathcal{F} = \{A|H, B|K, (B|K)|_{df}(A|H)\}$, with $(x, y, z) \in [0, 1]^3$ is coherent.

For instance, the assessment $(1, 1, 0)$ is coherent on $\mathcal{F}$ but it is not coherent on $\{A, B, B|A\}$.

Conjoined conditional as a suitable random quantity

Definition 7

Given any pair of conditional events $A|H$ and $B|K$, with $P(A|H) = x$, $P(B|K) = y$, we define their conjunction as [Gilio and Sanfilippo(2014)], (see also [McGee(1989), Flaminio et al.(2022a)Flaminio, Gilio, Godo, and Sanfilippo])

$$(A|H) \wedge_{gs} (B|K) = (ABHK + x\overline{H}BK + y\overline{K}AH)|(H \vee K) =$$
$$= \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}H \text{ is true or } \overline{B}K \text{ is true,} \\ x = P(A|H), & \text{if } \overline{H}BK \text{ is true,} \\ y = P(B|K), & \text{if } \overline{K}AH \text{ is true,} \\ z = \mathbb{P}[(A|H) \wedge_{gs} (B|K)], & \text{if } \overline{H}\,\overline{K} \text{ is true,} \end{cases}$$

In the betting framework you agree to pay $z = \mathbb{P}[(A|H) \wedge_{gs} (B|K)]$ with the proviso that you will receive:

- 1, if all conditional events are true;
- 0, if at least one of the conditional events is false;
- the probability of that conditional event which is void, if one conditional event is void and the other one is true;
- the quantity z that you paid, if both conditional events are void.

The structure $\Box|\bigcirc = \Box \wedge \bigcirc + \mathbb{P}(\Box|\bigcirc)\overline{\bigcirc}$

Given two conditional events $A|H$ and $B|K$, the iterated conditional $(B|K)|_{gs}(A|H)$ is defined as the random quantity ([Gilio and Sanfilippo(2013)])

$$(B|K)|_{gs}(A|H) = (A|H) \wedge_{gs} (B|K) + \mu_{gs}(\overline{A}|H),$$

where $\mu_{gs} = \mathbb{P}[(B|K)|_{gs}(A|H)]$. This definition exploits the structure

$$\Box|\bigcirc = \Box \wedge \bigcirc + \mathbb{P}(\Box|\bigcirc)\overline{\bigcirc}.$$

Then, by using the previous structure, for each trivalent logic we define the iterated conditional $(B|K)|_i(A|H)$ as

$$(B|K)|_i(A|H) = (A|H) \wedge_i (B|K) + \mu_i(\overline{A}|H), \;\; i \in \{K, L, B, S\}, \tag{3}$$

where $\mu_i = \mathbb{P}[(B|K)|_i(A|H)]$. For each $i \in \{K, L, B, S, gs\}$

- **(P1) is satisfied:** $[(A|H) \wedge (B|K)]|_i(A|H) = (B|K)|_i(A|H)$;
- **(P2) is satisfied:** $(A|H) \wedge_i (B|K) \leq (B|K)|_i(A|H)$ and hence $P[(A|H) \wedge_i (B|K)] \leq \mathbb{P}[(B|K)|_i(A|H)], \; i \in \{K, L, B, S, gs\}$.

Property P3

	$(B\|K)\|_K(A\|H)$	$(B\|K)\|_L(A\|H)$	$(B\|K)\|_B(A\|H)$	$(B\|K)\|_S(A\|H)$	$(B\|K)\|_{gs}(A\|H)$
$AHBK$	1	1	1	1	1
$AH\overline{B}K$	0	0	0	0	0
$AH\overline{K}$	$x\mu_K$	$x\mu_L$	$x\mu_B$	1	y
$\overline{A}HBK$	μ_K	μ_L	μ_B	μ_S	μ_{gs}
$\overline{A}H\overline{B}K$	μ_K	μ_L	μ_B	μ_S	μ_{gs}
$\overline{A}H\,\overline{K}$	μ_K	μ_L	$\mu_B(1+x)$	μ_S	μ_{gs}
$\overline{H}BK$	μ_K	μ_L	μ_B	$1+\ \mu_S(1-x)$	$x+\mu_{gs}(1-x)$
$\overline{H}\,\overline{B}K$	$\mu_K(1-x)$	$\mu_L(1-x)$	μ_B	$\mu_S(1-x)$	$\mu_{gs}(1-x)$
$\overline{H}\,\overline{K}$	μ_K	$\mu_L(1-x)$	μ_B	μ_S	μ_{gs}

For the linearity of prevision, we have that

$$\mathbb{P}((A|H) \wedge_i (B|K)) = \mathbb{P}((B|K)|_i(A|H))P(A|H), \;\; i \in \{K, L, B, S, gs\},$$

that is (P3) is satisfied. Moreover, when $P(A|H) > 0$, it follow that

$$\mathbb{P}[(B|K)|_i(A|H)] = \frac{\mathbb{P}((A|H) \wedge_i (B|K))}{P(A|H)}.$$

Generalized Bayes' Theorem I

We recall the Bayes formula

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\overline{B})P(\overline{B})}, \quad \text{if } P(A) > 0.$$

In our case, when $P(A|H) > 0$, as

$$\mathbb{P}[(B|K) \wedge_i (A|H)] = \mathbb{P}[(B|K)|_i(A|H)]P(A|H) = \mathbb{P}[(A|H)|_i(B|K)]P(B|K),$$

it holds that

$$\mathbb{P}[(B|K)|_i(A|H)] = \frac{\mathbb{P}[(A|H)|_i(B|K)]P(B|K)}{P(A|H)}, i \in \{K, L, B, S, gs\}.$$

It is well known that given two events A and B, it holds that $A = AB \vee A\overline{B}$, and hence $P(A) = P(AB) + P(A\overline{B}) = P(A|B)P(B) + P(A|\overline{B})P(B)$. However, when A, B are replaced by the conditional events $A|H, B|K$, respectively, it holds that

$$[(A|H) \wedge_i (B|K)] \vee_i [(A|H) \wedge_i (\overline{B}|K)] \neq A|H, \quad i \in \{K, L, B, S\}$$

Generalized Bayes' Theorem II

and only $[(A|H) \wedge_{gs} (B|K)] \vee_{gs} [(A|H) \wedge_{gs} (\overline{B}|K)] = A|H$.

Then, for each $i \in \{K, L, B, S\}$, $P(A|H)$ cannot be decomposed as $\mathbb{P}((A|H) \wedge_i (B|K)) + \mathbb{P}((A|H) \wedge_i (\overline{B}|K)) =$ $\mathbb{P}((A|H)|_i(B|K))P(B|K) + \mathbb{P}((A|H)|_i(\overline{B}|K))P(\overline{B}|K)$. Therefore, the following version of Bayes' formula

$$\mathbb{P}[(B|K)|_i(A|H)] = \frac{\mathbb{P}((A|H)|_i(B|K))P(B|K)}{\mathbb{P}((A|H)|_i(B|K))P(B|K) + \mathbb{P}((A|H)|_i(\overline{B}|K))P(\overline{B}|K)}$$

only holds for $|_{gs}$ and does not hold for $i \in \{K, L, B, S\}$.

Property P4: $|_K$, $|_L$

Concerning the pair $(\wedge_K, |_K)$, we have the following result

Theorem 8

Let A, B, H, K be any logically independent events. The set Π of all the coherent assessment (x, y, z, μ) on the family
$\mathcal{F} = \{A|H, B|K, (A|H) \wedge_K (B|K), (B|K)|_K(A|H)\}$ is $\Pi = \Pi' \cup \Pi''$, where
$\Pi' = \{(x, y, z, \mu) : x \in (0, 1], y \in [0, 1], z \in [z', z''], \mu = \frac{z}{x}\}$ with $z' = 0$,
$z'' = min\{x, y\}$, and $\Pi'' = \{(0, y, 0, \mu) : (y, \mu) \in [0, 1]^2\}$.

Based on Theorem 8, as the assessment $(1, 1, 0, 0)$ on $\{A|H, B|K, (A|H) \wedge_K (B|K), (B|K)|_K(A|H)\}$ is coherent, it follows that **(P4)** is not satisfied by $|_K$ (indeed $\mu = 0 < \mu' = \frac{\max\{1+1-1,0\}}{1}$=1).

In addition, it can be easily shown that statement of Theorem 8 also holds when $(\wedge_K, |_K)$ is replaced by $(\wedge_L, |_L)$. Then, also $|_L$ does not satisfy property **(P4)**.

Example with $|_K$ I

We show a counterintuitive aspect of the iterated conditional $|_K$.

- A **double-headed** coin is going to be either tossed or spun.

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- Consider the events H =“the coin is tossed”, $\overline{H}$ =“the coin is spun”, and A =“the coin comes up heads”.
- the two conditionals “if the coin is tossed it will come up heads” and “if the coin is spun it will come up heads”, have probability 1, i.e. $P(A|H) = P(A|\overline{H}) = 1$.
- However, as the coin cannot be both tossed and spun at the same time ($H\overline{H} = \emptyset$), it follows that $P[(A|H) \wedge_K (A|\overline{H})] = P(AH\overline{H}|AH\overline{H} \vee \overline{A}H \vee \overline{A}\,\overline{H}) = P(\emptyset|\overline{A}) = 0$ and hence $\mathbb{P}[(A|\overline{H})|_K(A|H)] = \frac{P[(A|H)\wedge_{KL}(A|\overline{H})]}{P(A|H)} = 0$.

Example with $|_K$ II

- Hence, the iterated conditional
  "**if** *the coin will come up heads, when it is tossed,* **then** *the coin will come up heads, when the coin is spun*"
  has, counterintuitively, a probability of 0, even if both conditionals, "if the coin is tossed it will come up heads" and "if the coin is spun it will come up heads", have probability 1.

Property P4: $|_B, \ |_S$

Concerning $|_B$, we observe that the assessment $(x, y, 1)$ on $\{A|H, B|K, (A|H) \wedge_B (B|K)\}$ is coherent for every $(x, y) \in [0,1]^2$. Then, when $0 < x < 1$, as $\mu = \frac{z}{x}$, the extension $\mu = \frac{1}{x} > 1$ on $(B|K)|_B(A|H)$ is coherent. That is, it is coherent to assess $\mathbb{P}[(B|K)|_B(A|H)] > 1$ and hence property **(P4)** is not satisfied by $|_B$ (indeed $\mu > 1 = \mu'' = \frac{\min\{1,1\}}{1} = 1$).

Likewise, we observe that the assessment $(x, 1, 1)$ on $\{A|H, B|K, (A|H) \wedge_S (B|K)\}$ is coherent for every $x \in [0,1]$. Then, when $0 < x < 1$, as $\mu = \frac{z}{x}$, the extension $\mu = \frac{1}{x} > 1$ on $(B|K)|_S(A|H)$ is coherent. That is, it is coherent to assess $\mathbb{P}[(B|K)|_S(A|H)] > 1$ and hence property **(P4)** is not satisfied by $|_S$.

Property P4: $|_{gs}$

Finally, differently from the other iterated conditionals, we recall that $|_{gs}$ satisfies **(P4)** ([Sanfilippo et al.(2018)Sanfilippo, Pfeifer, Over, and Gilio]). Indeed, given a coherent assessment (x, y) on $\{A|H, B|K\}$, under logical independence, for the iterated conditional $(B|K)|_{gs}(A|H)$ the extension $\mu = \mathbb{P}((B|K)|_{gs}(A|H))$ is coherent if and only if $\mu \in [\mu', \mu'']$, where

$$\mu' = \begin{cases} \frac{\max\{x+y-1,0\}}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \quad , \quad \mu'' = \begin{cases} \frac{\min\{x,y\}}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Therefore, $|_{gs}$ is the only iterated conditional which satisfies all the selected properties **(P1)-(P4)**.

Conclusions

- We recalled different notions of conjunction among conditional events such that the result of conjunction is still a conditional event.
- We studied Calabrese's and de Finetti's iterated conditional, we computed the lower and upper bounds and we showed that some basic properties are not satisfied;
- For each trivalent logic, we introduced the iterated conditional $(|_K, |_L, |_B, |_S)$ defined by exploiting the same structure used in order to define $|_{gs}$;
- We observed that, even if the compound prevision theorem is satisfied by all of them, only the iterated conditional $|_{gs}$ satisfies all the basic properties.

Thank you for your attention!

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Conditional Probability assessments, coherence and betting scheme

Let P be a probability function defined on an arbitrary family $\mathcal{K}$ of conditional events, consider $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\} \subseteq \mathcal{K}$ and $\mathcal{P} = (p_1, \dots, p_n)$, where $p_i = P(E_i|H_i)$, on $\mathcal{F}$. To the pair $(\mathcal{F}, \mathcal{P})$, we associate the random gain $G = \sum_{i=1}^{n} s_i H_i(E_i - p_i)$ and we denote by $\mathcal{G}_{\mathcal{H}_n}$ the set of values of G restricted to $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$.

Definition 9

The function P defined on $\mathcal{K}$ is coherent if and only if, $\forall n \geq 1$, $\forall\, s_1, \dots, s_n$, $\forall\, \mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\} \subseteq \mathcal{K}$, it holds that: $min\, \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq max\, \mathcal{G}_{\mathcal{H}_n}$.

In other words, *in any finite combination of n bets, after discarding the case where all the bets are called off, it does not happen that the values of the random gain are neither all positive nor all negative.*

Intervals of coherent extensions on $(A|H) \wedge_i (B|K)$ [Sanfilippo(2018)]

We recall that, by setting $P(A) = x, P(B) = y$, coherence requires that $P(AB) \in [\max\{x + y - 1, 0\}, \min\{x, y\}]$ (Fréchet-Hoeffding bounds).

Logical operations	Intervals of coherent extensions
$\wedge_K$	$[0,\ \min\{x, y\}]$
$\wedge_L$	$[0,\ \min\{x, y\}]$
$\wedge_B$	$[0,\ 1]$
$\wedge_S$	$\left[\max\{x + y - 1, 0\},\ \begin{cases} \frac{x+y-2xy}{1-xy}, & \text{if } (x, y) \neq (1, 1) \\ 1, & \text{if } (x, y) = (1, 1) \end{cases}\right]$
$\wedge_{gs}$	$[\max\{x + y - 1, 0\}, \min\{x, y\}]$

Table: Intervals of coherent extensions of the assessment (x, y) on $\{A|H, B|K\}$ to their conjunctions $\wedge_K$, $\wedge_L$, $\wedge_B$, $\wedge_S$,$\wedge_{gs}$.

Logical operations of conditionals and conditional random quantities

By setting $P(A|H) = x$, $P(B|K) = y$, $P[(A|H) \wedge_i (B|K)] = z_i$, $i \in \{K, L, B, S\}$ the conjunctions $(A|H) \wedge_i (B|K)$, $i \in \{K, L, B, S\}$ can be also looked at as random quantities.

	$A\|H$	$B\|K$	$\wedge_K$	$\wedge_L$	$\wedge_B$	$\wedge_S$	$\wedge_{gs}$
$AHBK$	1	1	1	1	1	1	1
$AH\overline{B}K$	1	0	0	0	0	0	0
$AH\overline{K}$	1	y	z_K	z_L	z_B	1	y
$\overline{A}HBK$	0	1	0	0	0	0	0
$\overline{A}H\overline{B}K$	0	0	0	0	0	0	0
$\overline{A}H\,\overline{K}$	0	y	0	0	z_B	0	0
$\overline{H}BK$	x	1	z_K	z_L	z_B	1	x
$\overline{H}\,\overline{B}K$	x	0	0	0	z_B	0	0
$\overline{H}\,\overline{K}$	x	y	z_K	0	z_B	z_S	z_{gs}